Lecture 08

Physics-based Simulation and Articulated Rigid Bodies

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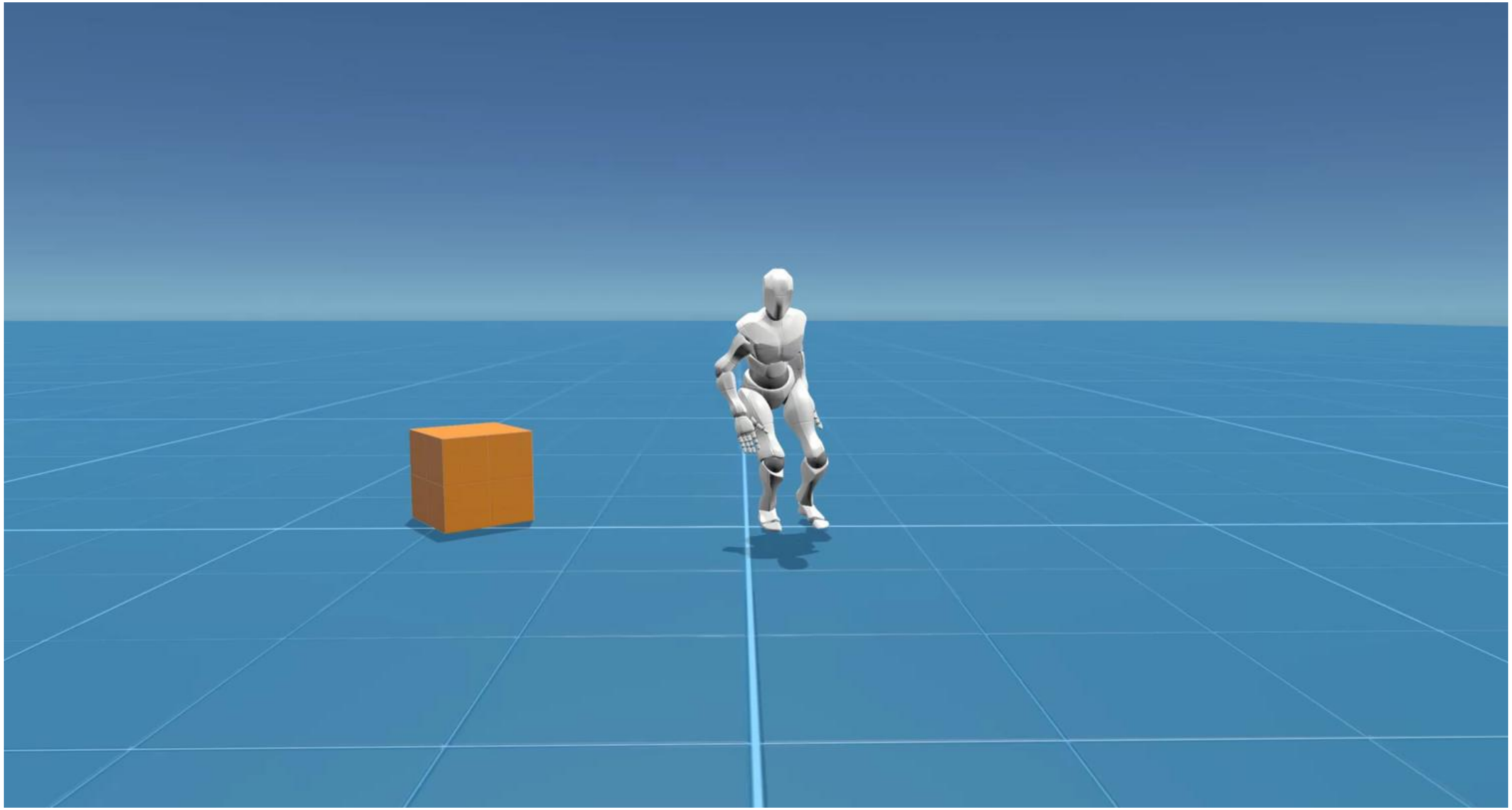
GAMES105 课程交流

VCL @ PKU

Problems of Kinematic Methods

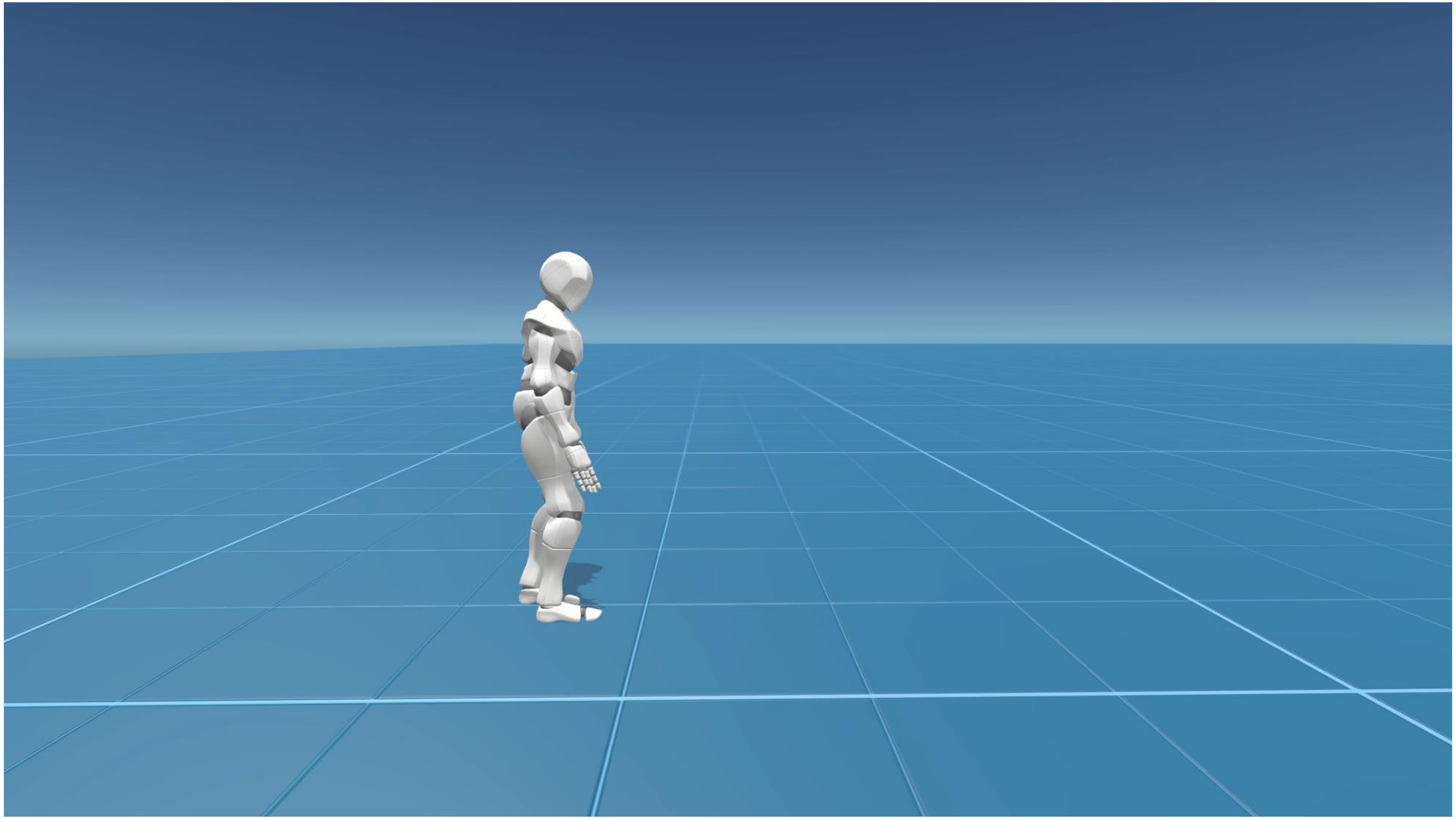
- Interaction with the environment

Physics-based Character Animation



[ControlVAE – Yao et al. 2022]

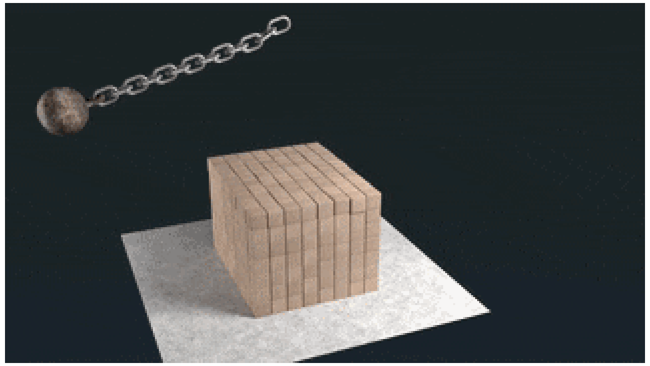
Physics-based Character Animation

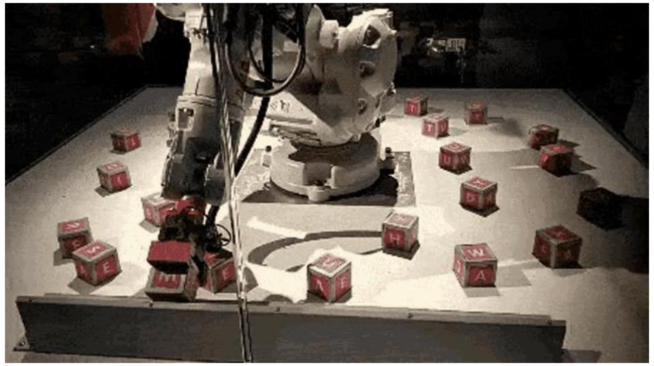


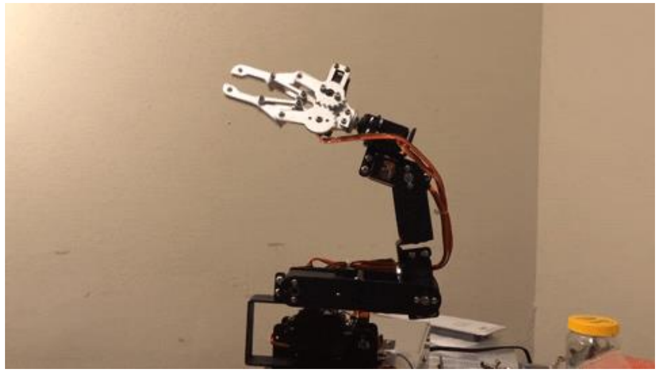
[ControlVAE – Yao et al. 2022]

Outline

- Simulation Basis
  - Numerical Integration: Euler methods
- Equations of Rigid Bodies
  - Rigid Body Kinematics
  - Newton-Euler equations
- Articulated Rigid Bodies
  - Joints and constraints
- Contact Models
  - Penalty-based contact
  - Constraint-based contact

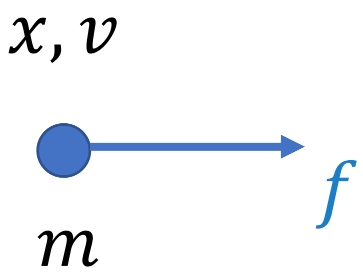






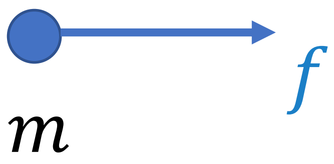
https://www.cs.cmu.edu/~baraff/sigcourse/

Dynamics of a Particle



Dynamics of a Particle

$$x(t=0)$$
$$v(t=0)$$



$$x(t=10)=?$$

Dynamics of a Particle

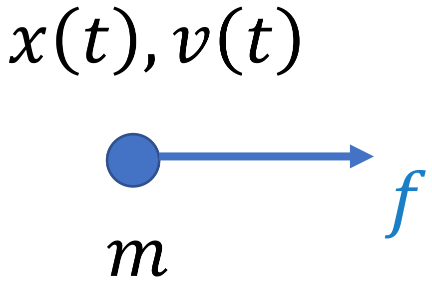
$x(t), v(t)$

f

m

$$f = ma$$

Dynamics of a Particle

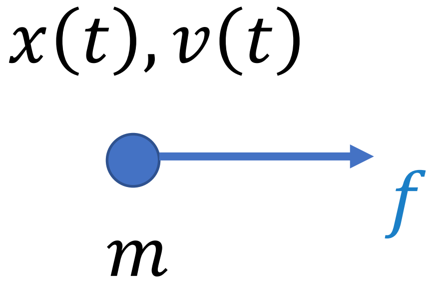


$$f = ma$$

$$a = \dot{v}$$

$$v = \dot{x}$$

Dynamics of a Particle



$$f = ma$$

$$a = \dot{v}$$

$$v = \dot{x}$$

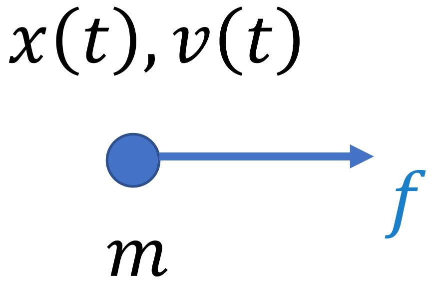
$\Rightarrow$

$$a = f/m$$

$$v = v_0 + \int_{t_0}^{t} a dt$$

$$x = x_0 + \int_{t_0}^{t} v dt$$

Dynamics of a Particle



$$f = ma$$

$$a = \dot{v}$$

$$v = \dot{x}$$

$\Rightarrow$

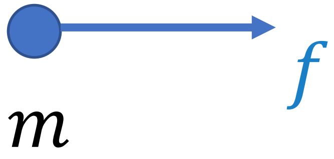
$$a = f/m$$

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

Dynamics of a Particle

$$x(t), v(t)$$



$$x(t = 10) = x_0 + 10v_0 + 50\frac{f}{m}$$

$$f = ma$$

$$a = \dot{v}$$

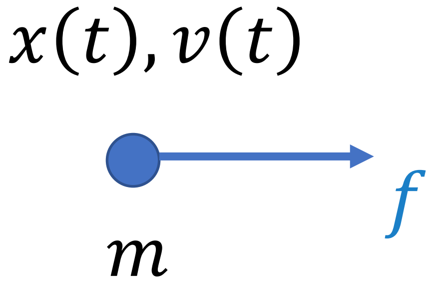
$$v = \dot{x}$$

$$a = f/m$$

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

Dynamics of a Particle



$x(t = 10) = ?$

$$f = ma$$

$$a = \dot{v}$$

$$v = \dot{x}$$

$$a = f(x, v, t)/m$$

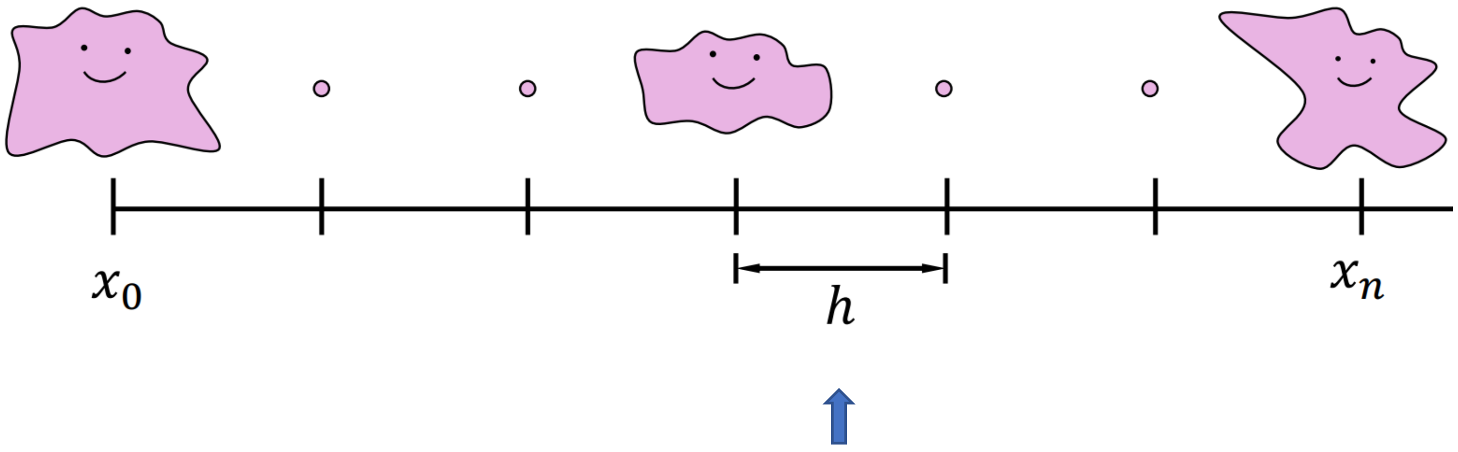
$$v = v_0 + \int_{t_0}^{t} a dt$$

$$x = x_0 + \int_{t_0}^{t} v dt$$

Temporal Discretization

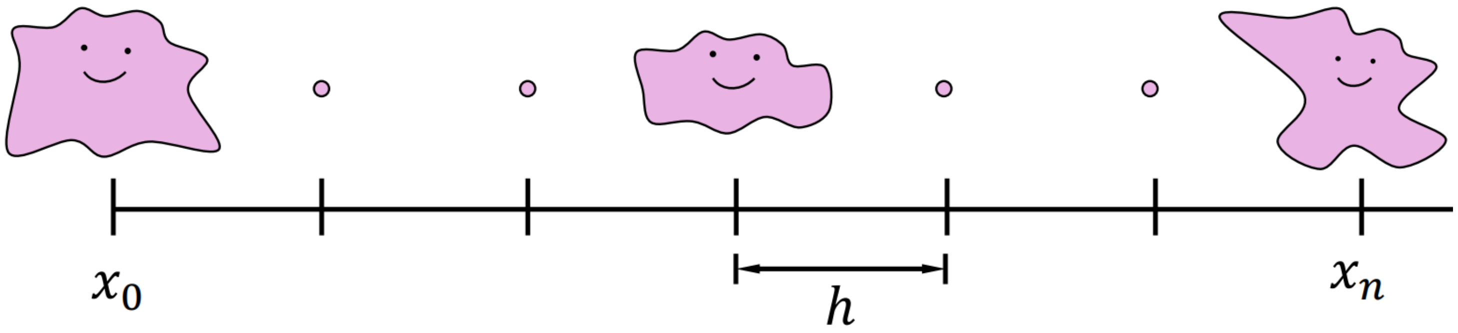
$$x = x(t)$$

$$x_n = x(t_n) \qquad t_n = nh$$



Simulation time step

Temporal Discretization



$$a = f(x, v, t)/m$$

$$v = v_0 + \int_{t_0}^{t} a dt$$

$$x = x_0 + \int_{t_0}^{t} v dt$$

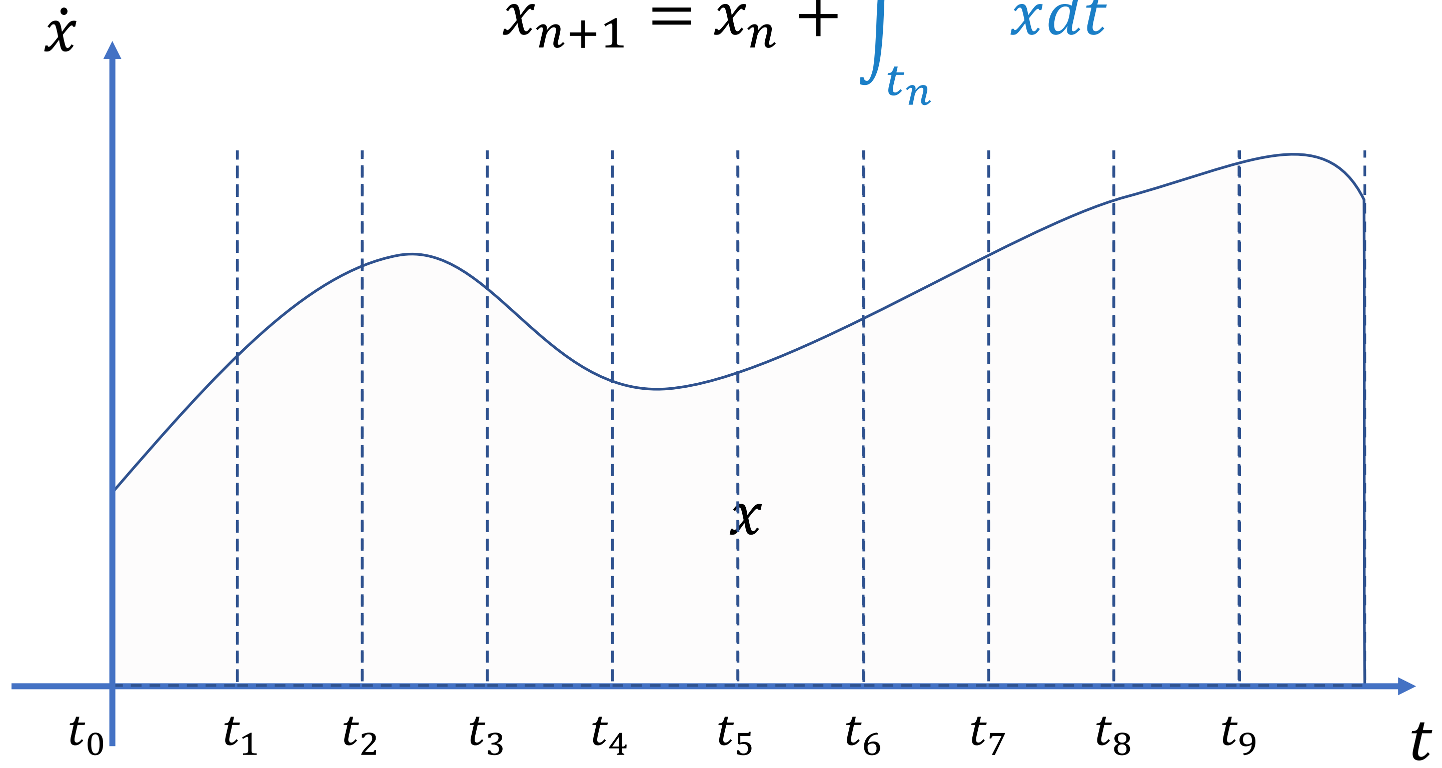
$$a = f(x, v, t)/m$$

$$v_{n+1} = v_n + \int_{t_n}^{t_{n+1}} a dt$$

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} \mathrm{v} dt$$

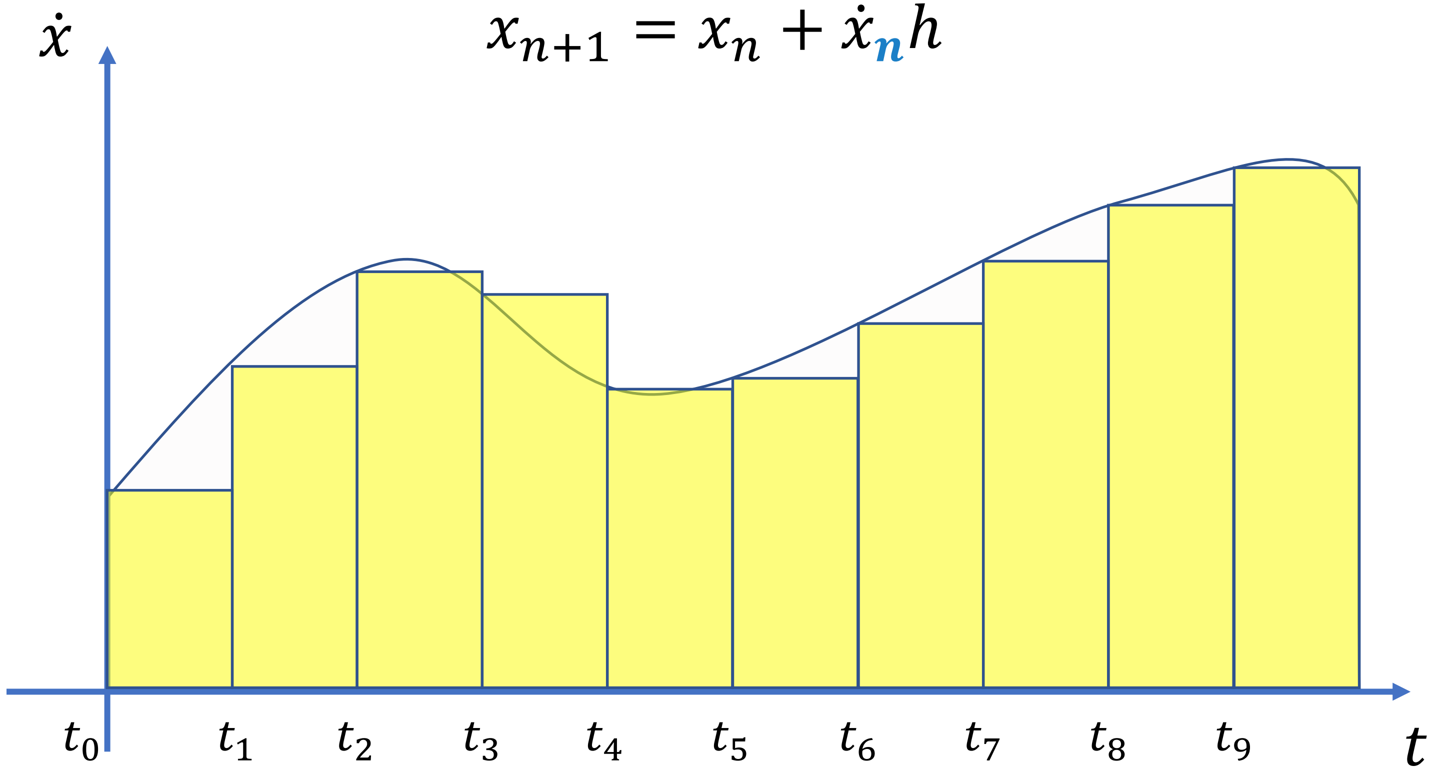
Numerical Integration

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} \dot{x} dt$$



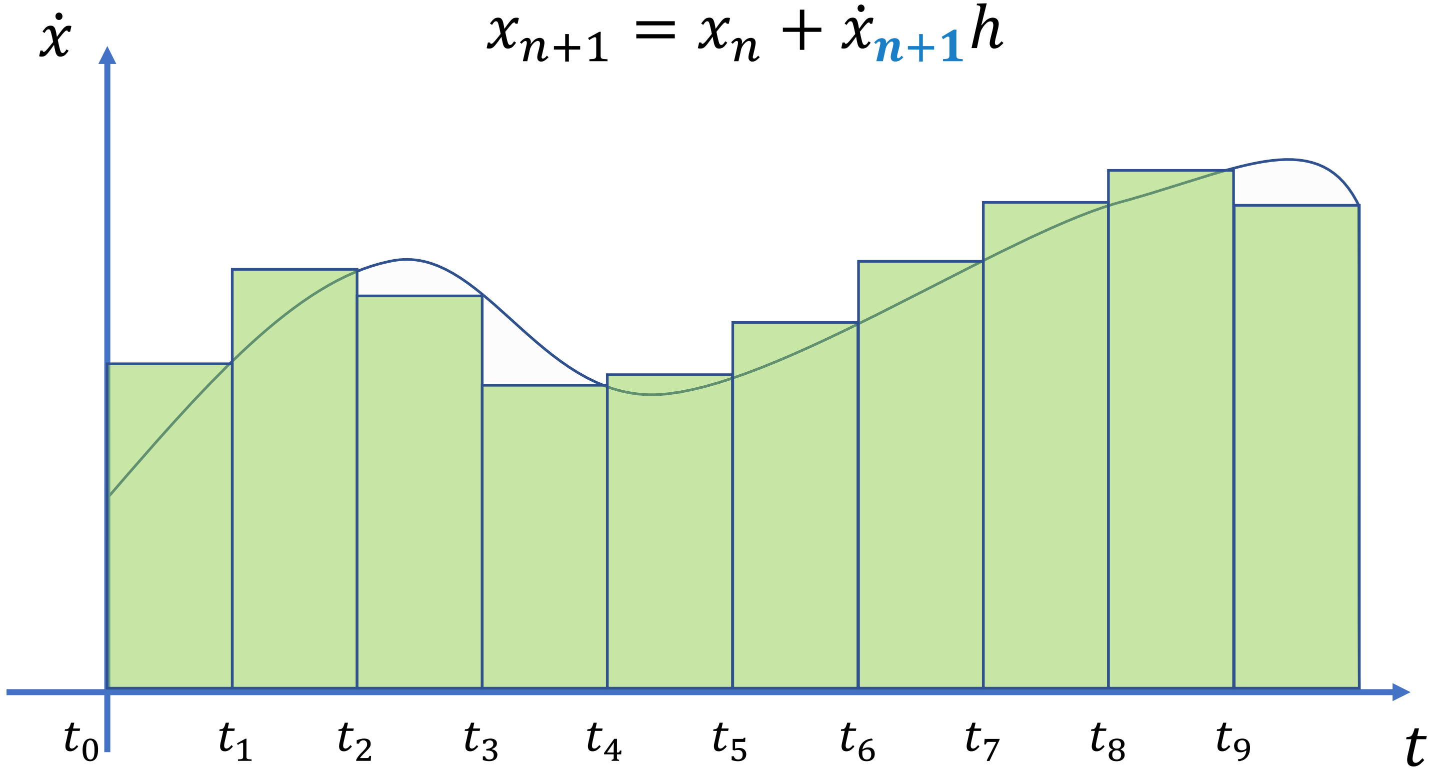
Numerical Integration

$$x_{n+1} = x_n + \dot{x}_n h$$



Numerical Integration

$$x_{n+1} = x_n + \dot{x}_{\boldsymbol{n+1}} h$$



Numerical Integration

- Explicit/Forward Euler Integration

$$v_{n+1} = v_n + a_{\boldsymbol{n}} h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n}} h$$

- Implicit/Backward Euler Integration

$$v_{n+1} = v_n + a_{\boldsymbol{n+1}} h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n+1}} h$$

Numerical Integration

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- Implicit/Backward Euler Integration

$$v_{n+1} = v_n + a_{\boldsymbol{n+1}} h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n+1}} h$$

Requires *"future"* information

Numerical Integration

- Explicit/Forward Euler Integration

$$v_{n+1} = v_n + a_{\boldsymbol{n}} h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n}} h$$

- Implicit/Backward Euler Integration

$$v_{n+1} = v_n + f(x_{n+1}, v_{n+1})h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n+1}} h$$

← Requires *"future"* information

Numerical Integration

- Explicit/Forward Euler Integration

$$v_{n+1} = v_n + a_{\boldsymbol{n}} h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n}} h$$

- Implicit/Backward Euler Integration

$$v_{n+1} = v_n + a_{\boldsymbol{n+1}} h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n+1}} h$$

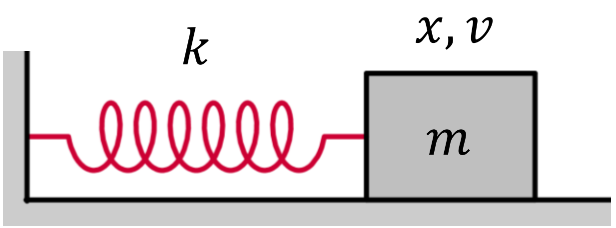
Requires *"future"* information

- Symplectic / Semi-implicit Euler Integration

$$v_{n+1} = v_n + a_{\boldsymbol{n}} h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n+1}} h$$

All information is current

Mass on a Spring



$$f = -kx$$

Explicit Euler Integration

$$v_{n+1} = v_n - \frac{kx_n}{m}h$$
$$x_{n+1} = x_n + v_n h$$

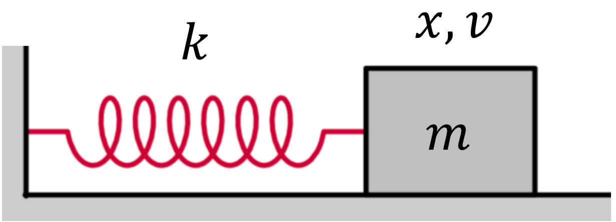
Semi-implicit Euler Integration

$$v_{n+1} = v_n - \frac{kx_n}{m}h$$
$$x_{n+1} = x_n + v_{n+1} h$$

Implicit Euler Integration

$$v_{n+1} = v_n - \frac{kx_{n+1}}{m}h$$
$$x_{n+1} = x_n + v_{n+1} h$$

Mass on a Spring



$$f = -kx$$
$$\hat{k} = k/m$$

Explicit Euler Integration

$$\begin{bmatrix} v_{n+1} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\hat{k}h \\ h & 1 \end{bmatrix} \begin{bmatrix} v_n \\ x_n \end{bmatrix}$$

Semi-implicit Euler Integration

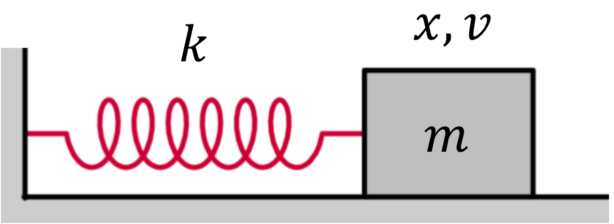
$$\begin{bmatrix} v_{n+1} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\hat{k}h \\ h & 1 - \hat{k}h^2 \end{bmatrix} \begin{bmatrix} v_n \\ x_n \end{bmatrix}$$

Implicit Euler Integration

$$\begin{bmatrix} 1 & \hat{k}h \\ -h & 1 \end{bmatrix} \begin{bmatrix} v_{n+1} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} v_n \\ x_n \end{bmatrix}$$

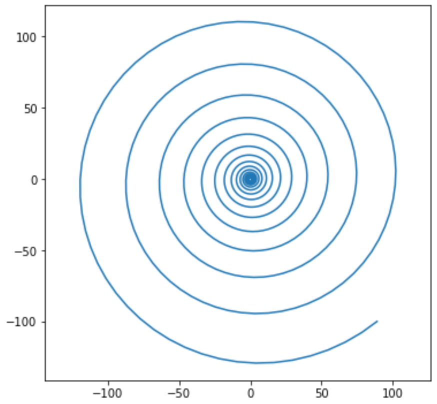
$$\begin{bmatrix} v_{n+1} \\ x_{n+1} \end{bmatrix} = \frac{1}{1 + \hat{k}h^2} \begin{bmatrix} 1 & -\hat{k}h \\ h & 1 \end{bmatrix} \begin{bmatrix} v_n \\ x_n \end{bmatrix}$$

Mass on a Spring

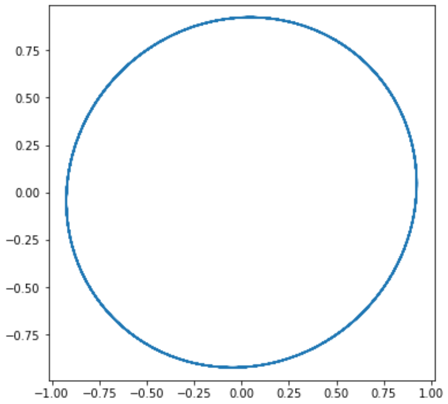


$$f = -kx$$

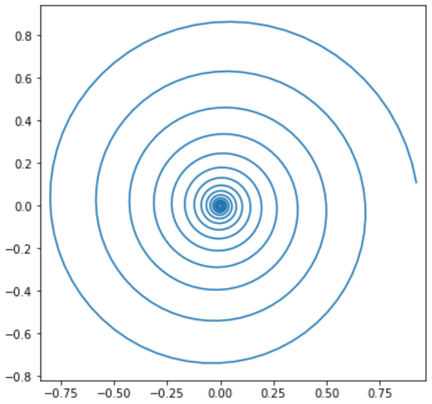
Explicit Euler Integration



Semi-implicit Euler Integration



Implicit Euler Integration



Numerical Integration

- Explicit/Forward Euler
  Symplectic/Semi-implicit Euler
  - Fast, no need to solve equations
  - Can be unstable under large time step

$$v_{n+1} = v_n + f(x_n, v_n)h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n}}h$$

$$v_{n+1} = v_n + f(x_n, v_n)h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n+1}}h$$

- Implicit/Backward Euler
  - Rock stable (unconditionally)
  - Slow, need to solve a large problem

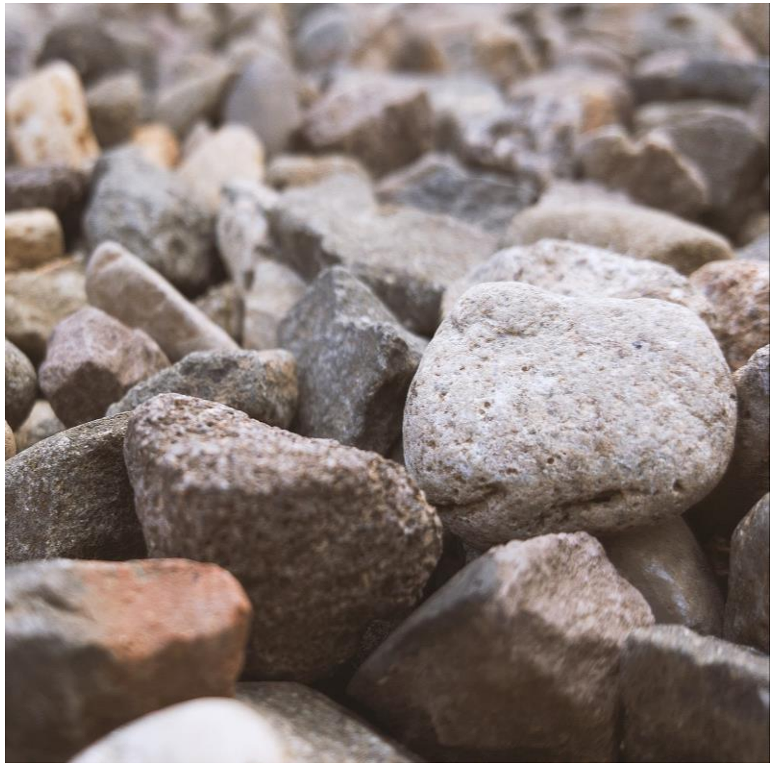
$$v_{n+1} = v_n + f(x_{n+1}, v_{n+1})h$$
$$x_{n+1} = x_n + v_{\boldsymbol{n+1}}h$$

More Advanced Integration

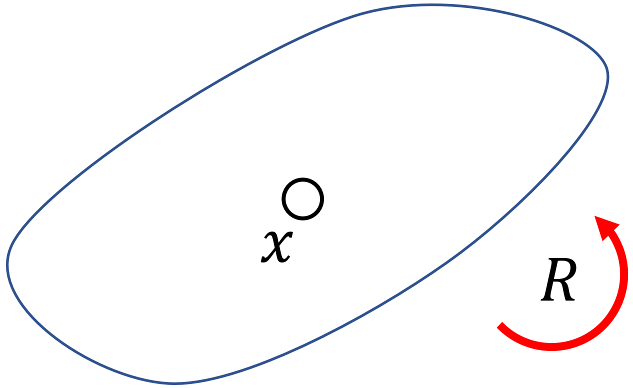
- Runge–Kutta methods
- Variational integration
- Position-based dynamics (PBD)
- ……

Rigid Bodies

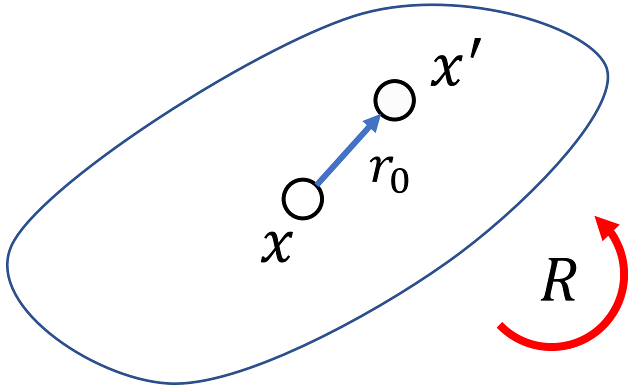
- They are rigid….



Position and Orientation

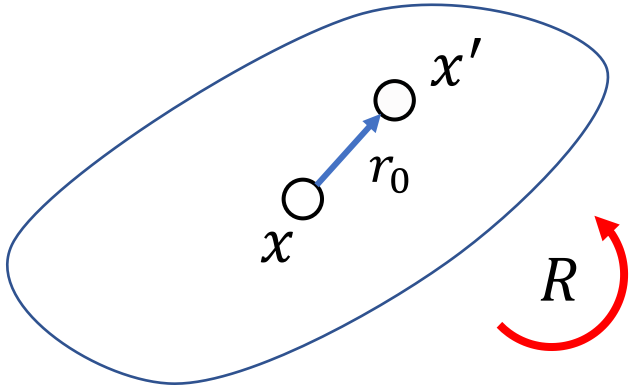


Position and Orientation



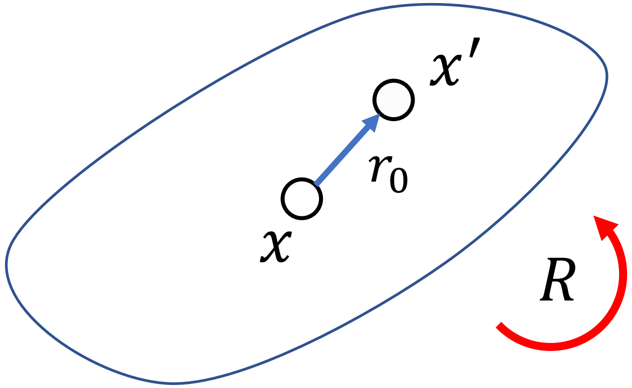
$$x' = x + Rr_0$$

Position and Orientation



$$x' = x + Rr_0 = x + r$$

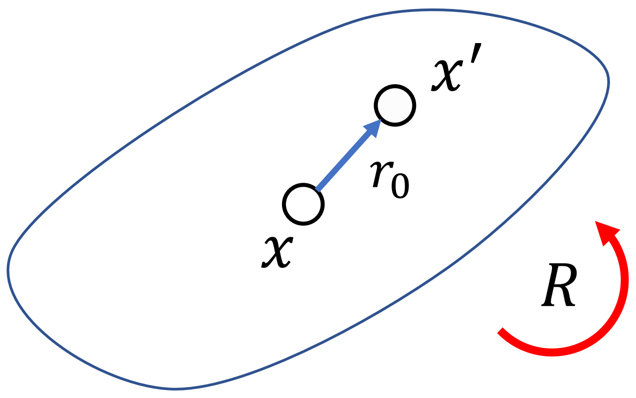
Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} = ?$$

Linear and Angular Velocity

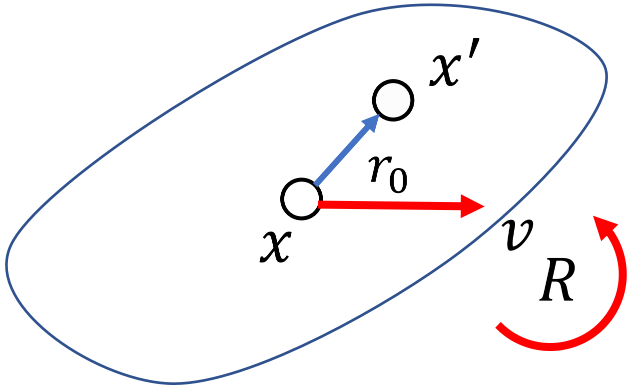


$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + \dot{R}r_0$$

$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} = ?$$

Linear and Angular Velocity



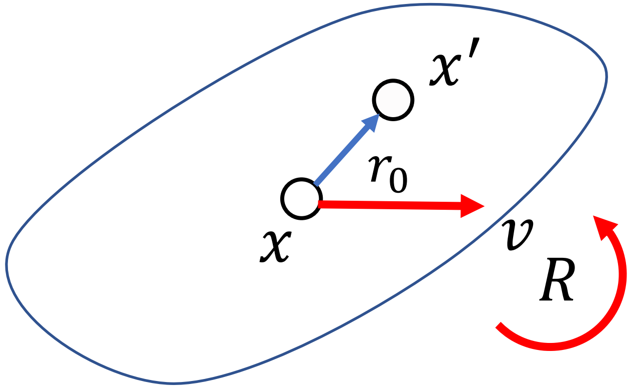
$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} = ?$$

$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + \dot{R}r_0$$

$$\dot{x} \Downarrow v$$

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

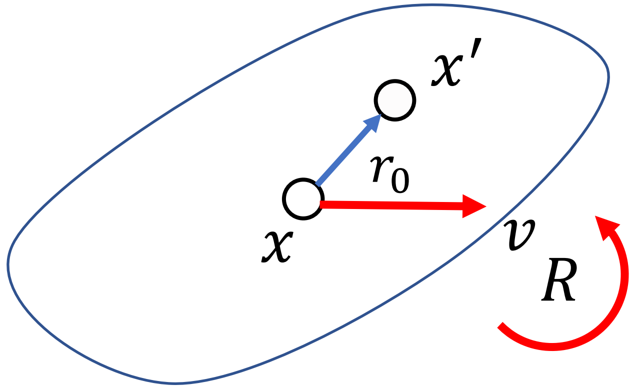
$$\frac{dx'}{dt} = ?$$

???

$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + \dot{R}r_0$$

v

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} = ?$$

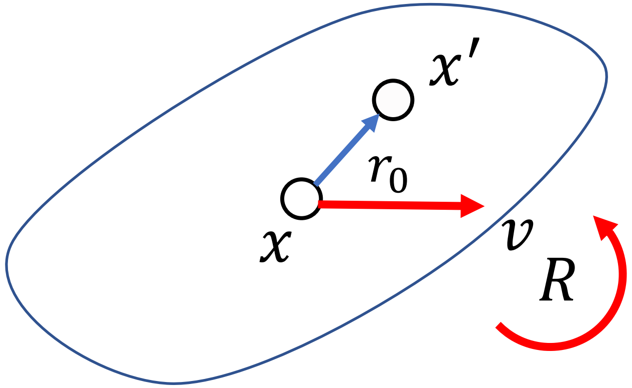
???

$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + \dot{R}r_0$$

v

$$RR^T = I$$

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} = ?$$

???

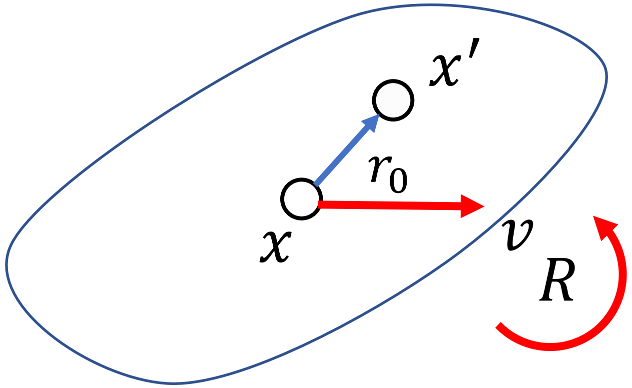
$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + \dot{R}r_0$$

v

$$RR^T = I$$

$$\frac{d(RR^T)}{dt} = 0$$

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} = ?$$

???

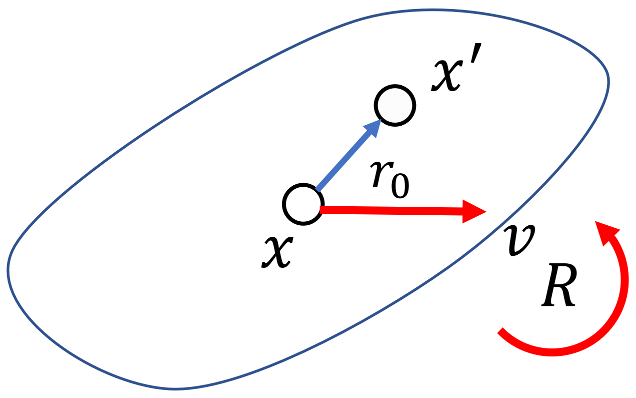
$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + \dot{R}r_0$$

$\dot{x} \rightarrow v$

$$RR^T = I$$

$$\dot{R}R^T + R\dot{R}^T = 0$$

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} =?$$

???

$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + \dot{R}r_0$$

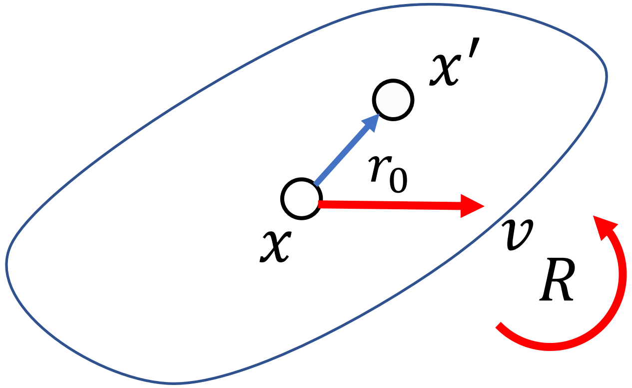
v

$$RR^T = I$$

$$\dot{R}R^T + \left(\dot{R}R^T\right)^T = 0$$

$\dot{R}R^T$ is a Skew-Symmetric Matrix

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} = ?$$

???

$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + \dot{R}r_0$$

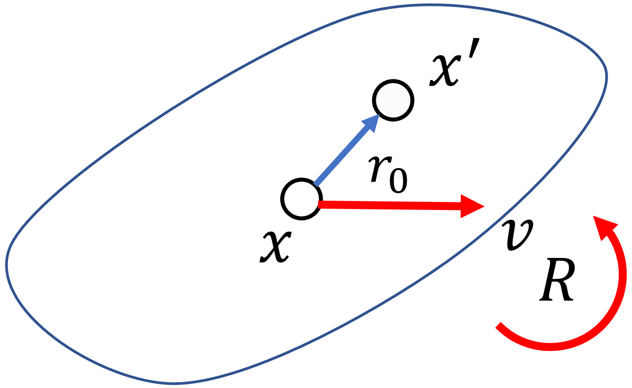
v

$$RR^T = I$$

$$\dot{R}R^T + \left(\dot{R}R^T\right)^T = 0$$

$$\dot{R}R^T = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = [\omega]_\times$$

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} = ?$$

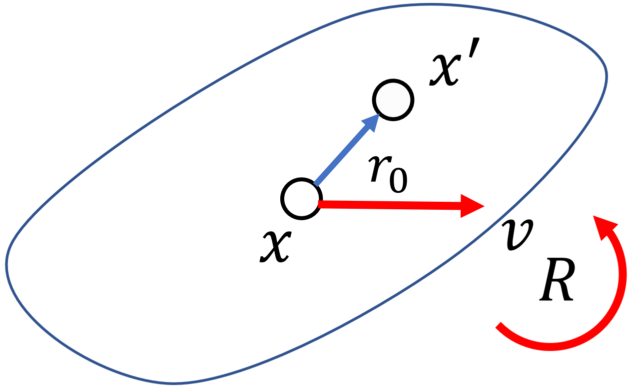
$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + [\omega]_\times R r_0$$

$$v$$

$$\dot{R} = [\omega]_\times R$$

$$\dot{R}R^T = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = [\omega]_\times$$

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$\frac{dx'}{dt} = ?$$

$$\frac{dx'}{dt} \Leftrightarrow \dot{x}' = \dot{x} + \omega \times (Rr_0)$$

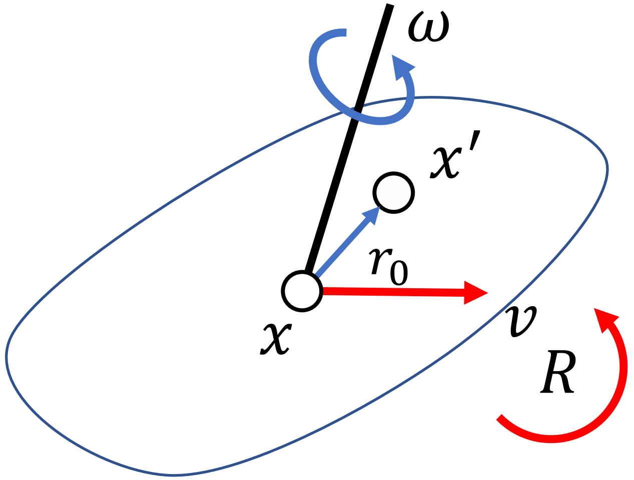
$\dot{x}$ ⇓ v

$$\dot{R} = [\omega]_\times R$$

⇑

$$\dot{R}R^T = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = [\omega]_\times$$

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$v' = v + \omega \times r$$

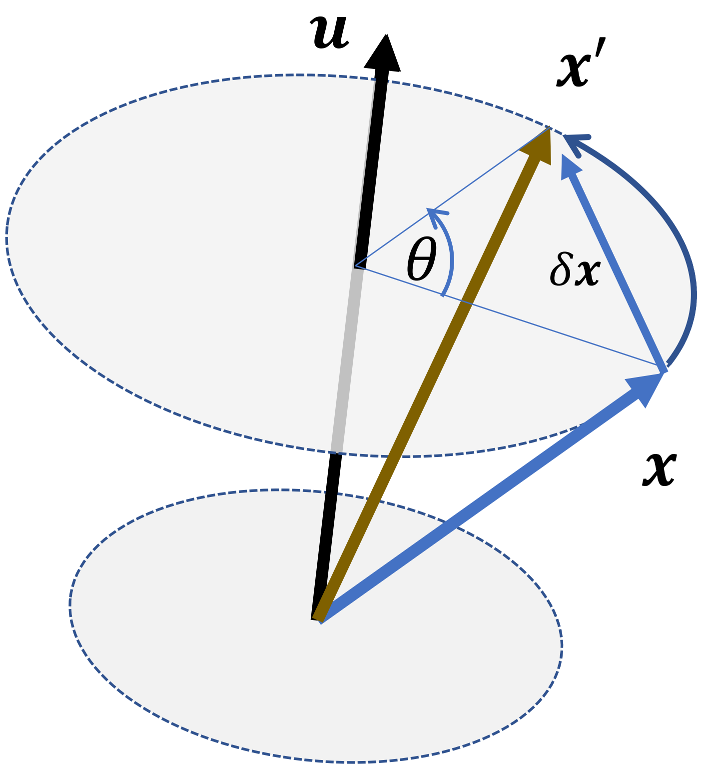
$$\dot{x} = v$$

$$\dot{R} = [\omega]_\times R$$

v: linear velocity

ω: angular velocity

Angular Velocity and Rotation Matrix



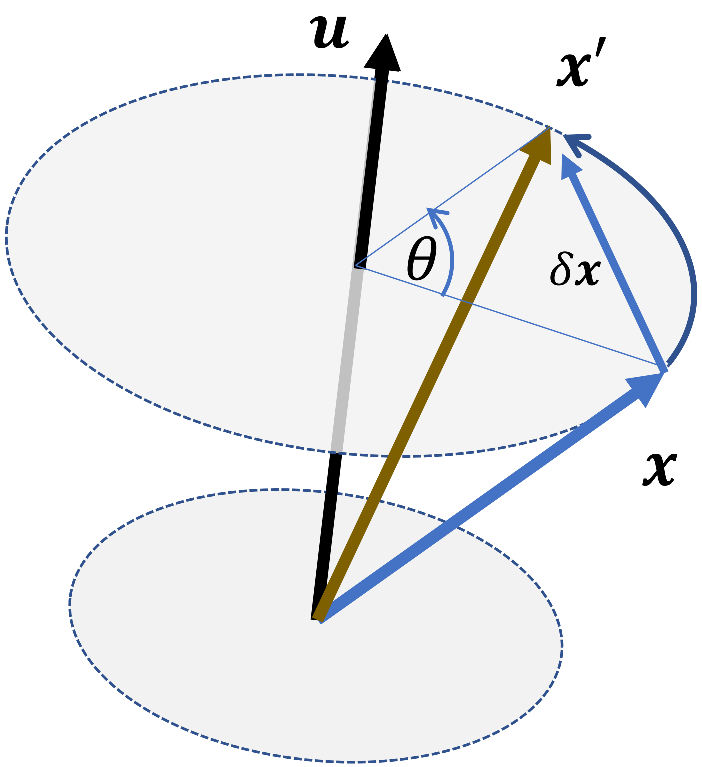
$$\|\boldsymbol{u}\| = \mathbf{1}$$

Rodrigues' rotation formula

$$\delta\boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{x}$$

$$= (\sin\theta)\,\boldsymbol{u}\times\boldsymbol{x} + (1-\cos\theta)\,\boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{x})$$

Angular Velocity and Rotation Matrix



$\|\boldsymbol{u}\| = \mathbf{1}$

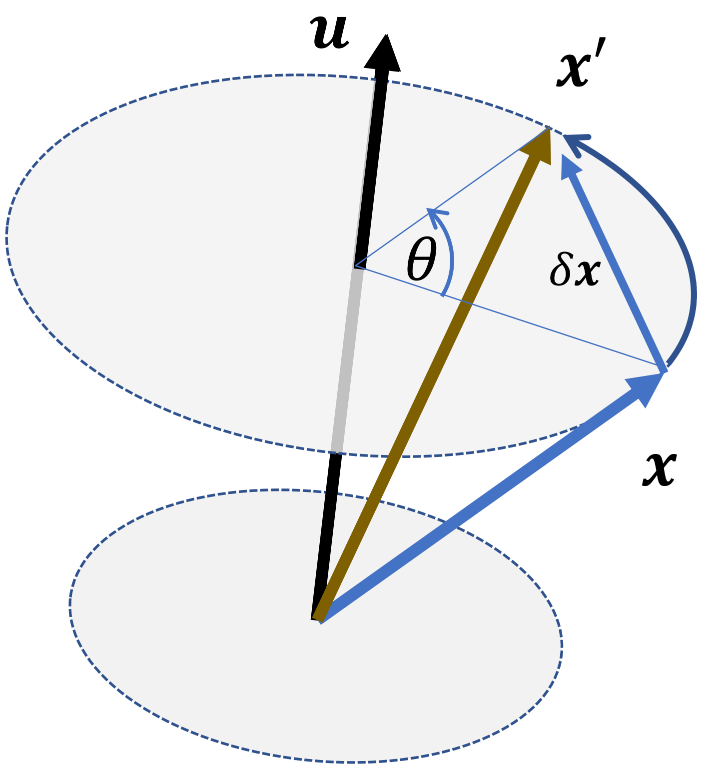
Rodrigues' rotation formula

$$\delta \boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{x}$$

$$= (\sin\theta)\, \boldsymbol{u} \times \boldsymbol{x} + (1 - \cos\theta)\, \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{x})$$

$$\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{dt} = \frac{d\boldsymbol{x}}{d\theta} \cdot \frac{d\theta}{dt} = \dot{\theta} \boldsymbol{u} \times \boldsymbol{x}$$

Angular Velocity and Rotation Matrix



$\|\boldsymbol{u}\| = \mathbf{1}$

Rodrigues' rotation formula

$$\delta \boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{x}$$

$$= (\sin\theta)\, \boldsymbol{u} \times \boldsymbol{x} + (1 - \cos\theta)\, \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{x})$$

⇩

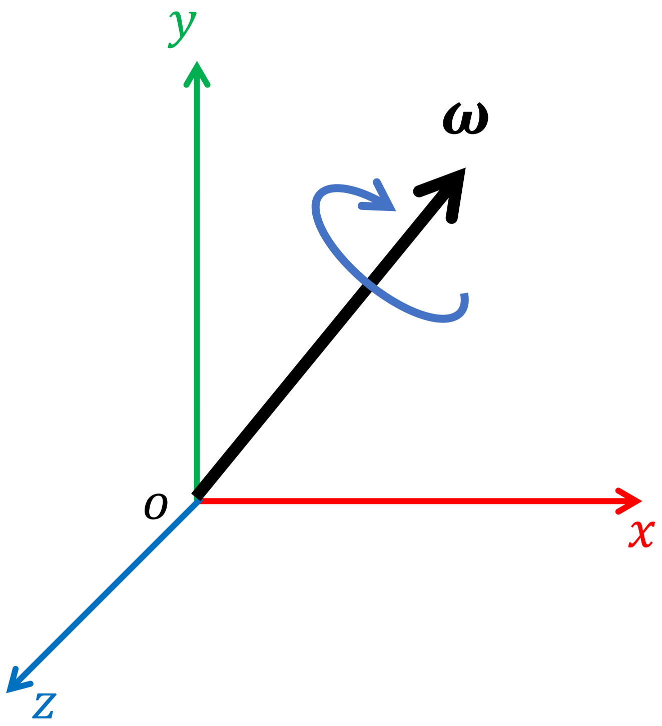
$$\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{dt} = \frac{d\boldsymbol{x}}{d\theta} \cdot \frac{d\theta}{dt} = \dot{\theta}\boldsymbol{u} \times \boldsymbol{x}$$

⇩

$$\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x}$$

Angular Velocity and Rotation Matrix

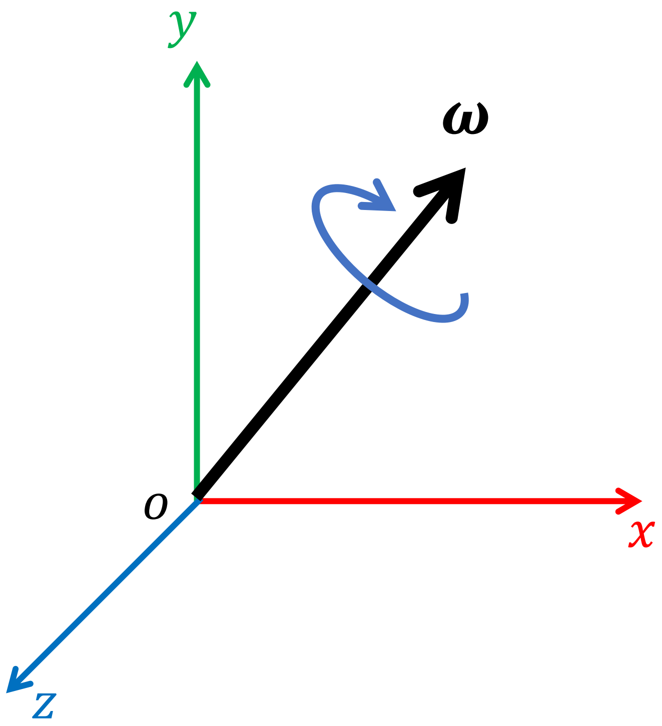
ω: angular velocity ⟺ $\dot{R} = [\omega]_{\times} R$



$$R = \begin{bmatrix} | & | & | \\ \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ | & | & | \end{bmatrix}$$

Angular Velocity and Rotation Matrix

ω: angular velocity $\Longleftrightarrow$ $\dot{R} = [\omega]_\times R$



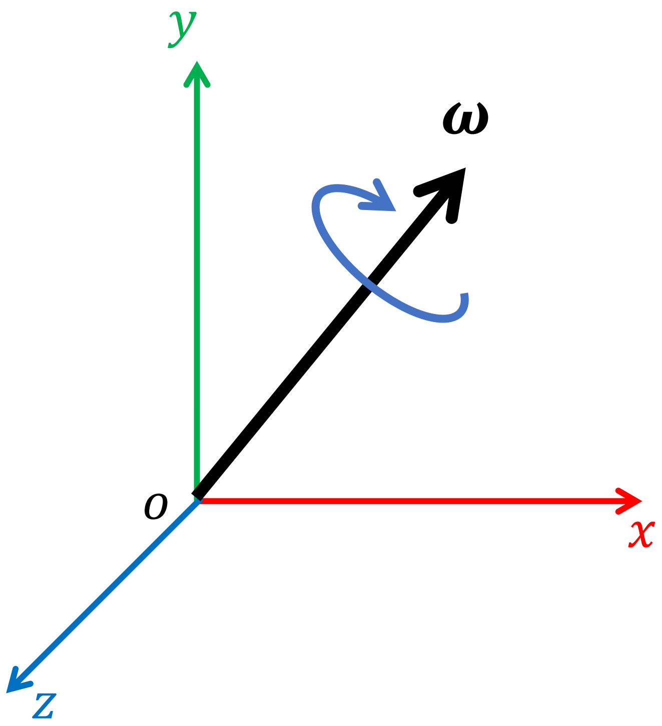
$$R = \begin{bmatrix} | & | & | \\ \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ | & | & | \end{bmatrix}$$

$\Downarrow$

$$\dot{\boldsymbol{e}}_x = \boldsymbol{\omega} \times \boldsymbol{e}_x$$
$$\dot{\boldsymbol{e}}_y = \boldsymbol{\omega} \times \boldsymbol{e}_y$$
$$\dot{\boldsymbol{e}}_z = \boldsymbol{\omega} \times \boldsymbol{e}_z$$

Angular Velocity and Rotation Matrix

ω: angular velocity $\quad \Longleftrightarrow \quad \dot{R} = [\omega]_\times R$

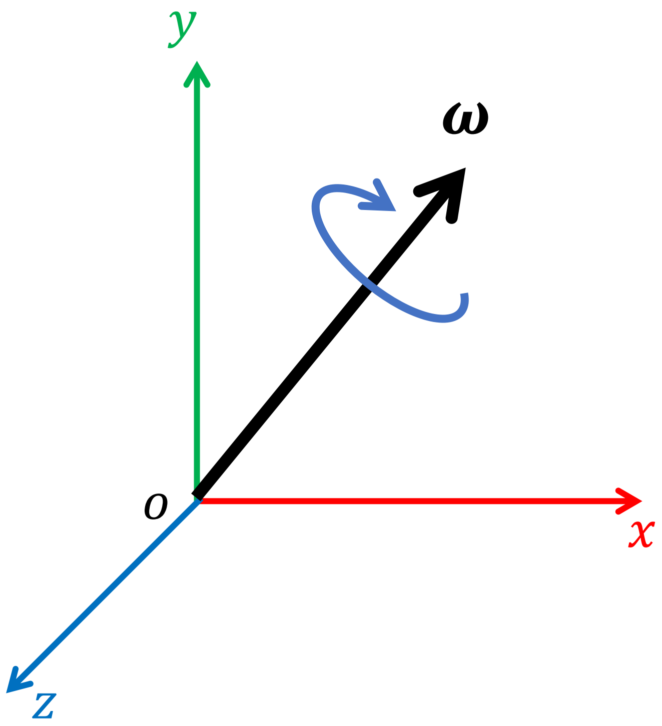


$$R = \begin{bmatrix} | & | & | \\ \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ | & | & | \end{bmatrix}$$

$$\dot{R} = \begin{bmatrix} | & | & | \\ \dot{\boldsymbol{e}}_x & \dot{\boldsymbol{e}}_y & \dot{\boldsymbol{e}}_z \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \boldsymbol{\omega} \times \dot{\boldsymbol{e}}_x & \boldsymbol{\omega} \times \dot{\boldsymbol{e}}_y & \boldsymbol{\omega} \times \dot{\boldsymbol{e}}_z \\ | & | & | \end{bmatrix}$$

Angular Velocity and Rotation Matrix

ω: angular velocity $\Longleftrightarrow$ $\dot{R} = [\omega]_\times R$

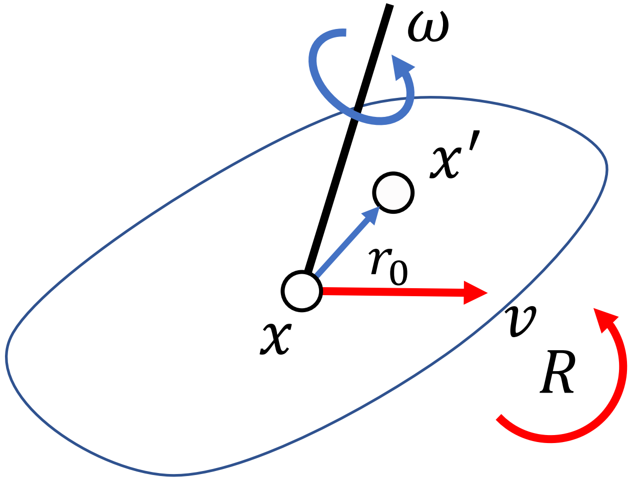


$$R = \begin{bmatrix} | & | & | \\ \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ | & | & | \end{bmatrix}$$

$$\Downarrow$$

$$\dot{R} = \begin{bmatrix} | & | & | \\ \dot{\boldsymbol{e}}_x & \dot{\boldsymbol{e}}_y & \dot{\boldsymbol{e}}_z \\ | & | & | \end{bmatrix} = [\boldsymbol{\omega}]_\times \begin{bmatrix} | & | & | \\ \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ | & | & | \end{bmatrix}$$

Linear and Angular Velocity



$$x' = x + Rr_0 = x + r$$

$$v' = v + \omega \times r$$

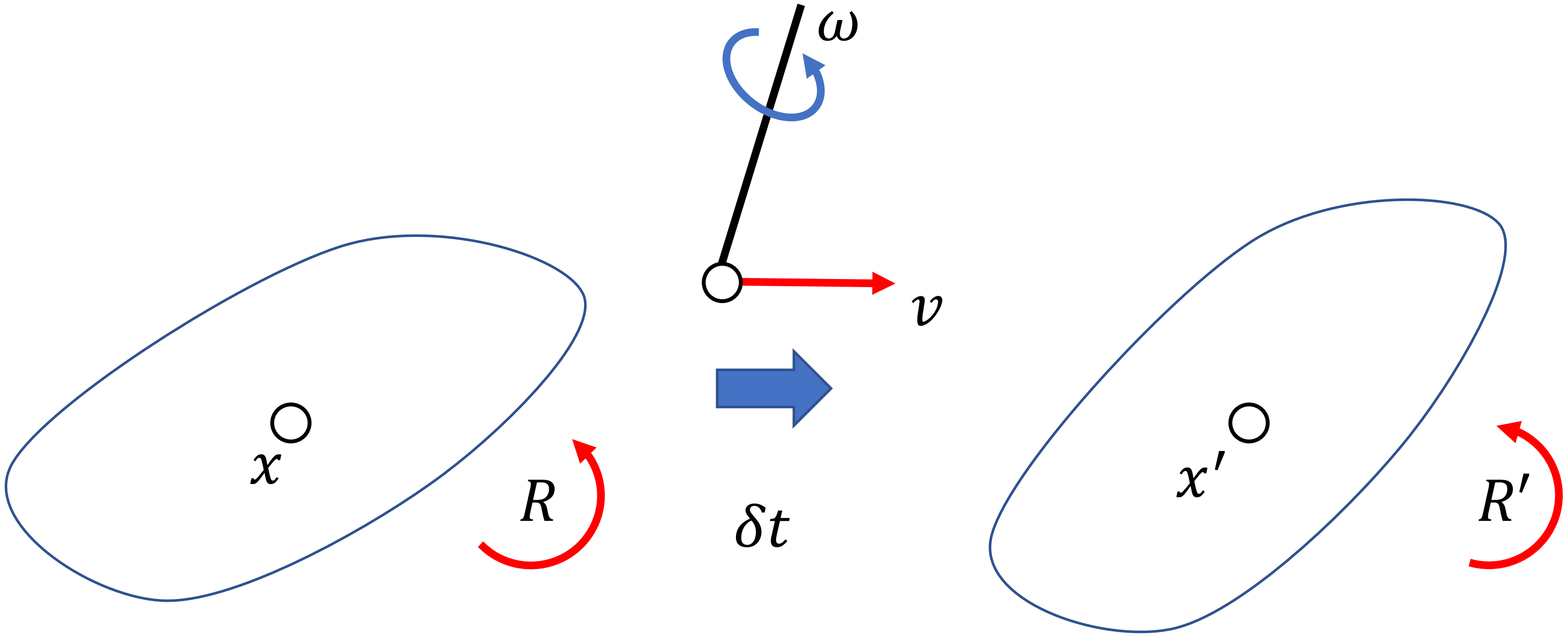
$$\dot{x} = v$$

$$\dot{R} = [\omega]_\times R$$

v: linear velocity

ω: angular velocity

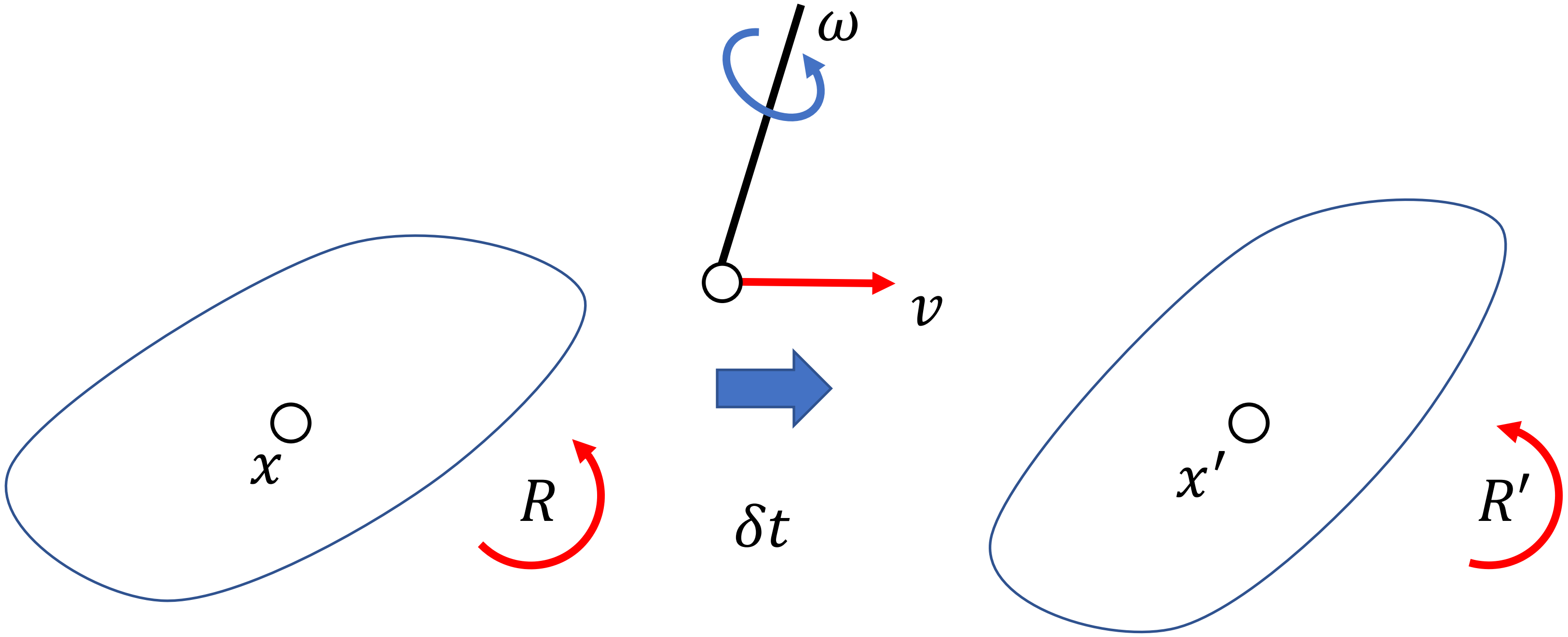
Numerical Integration



$$x' = ?$$

$$R' = ?$$

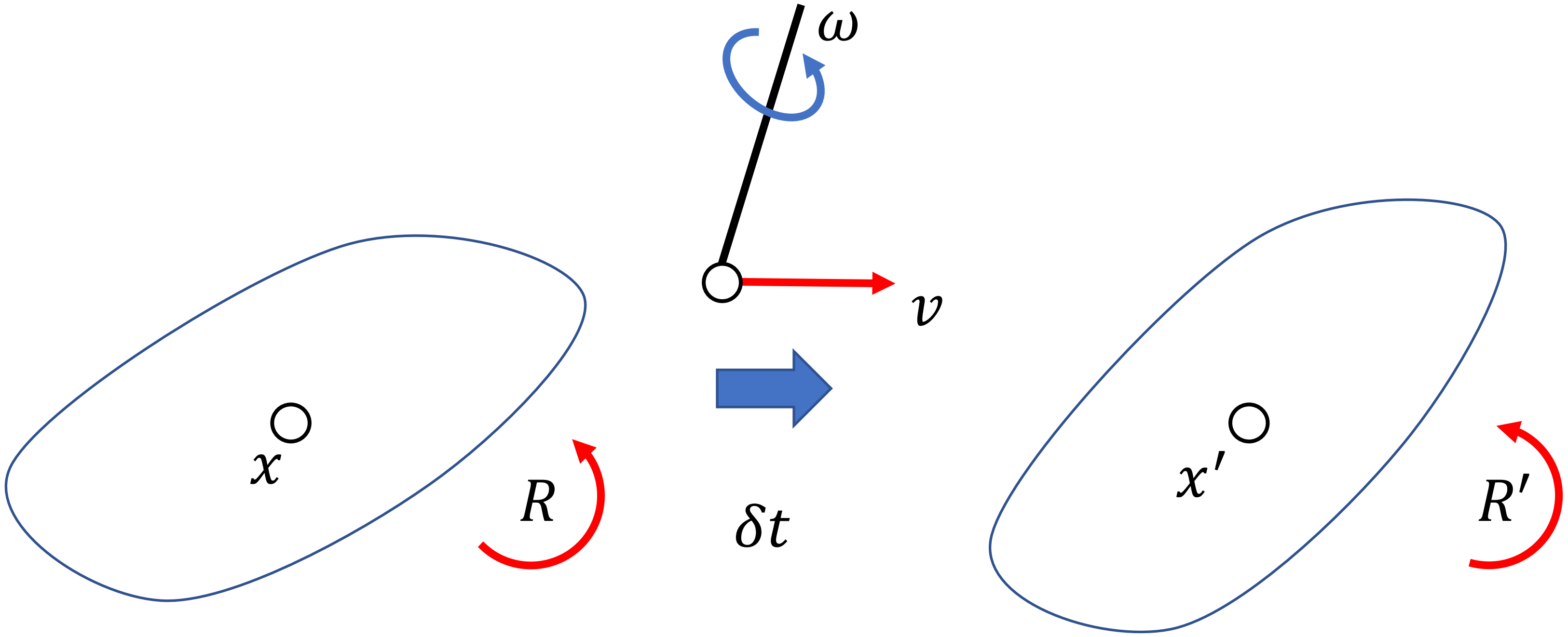
Numerical Integration



$$\dot{x} = v$$

$$\dot{R} = [\omega]_{\times} R$$

Numerical Integration

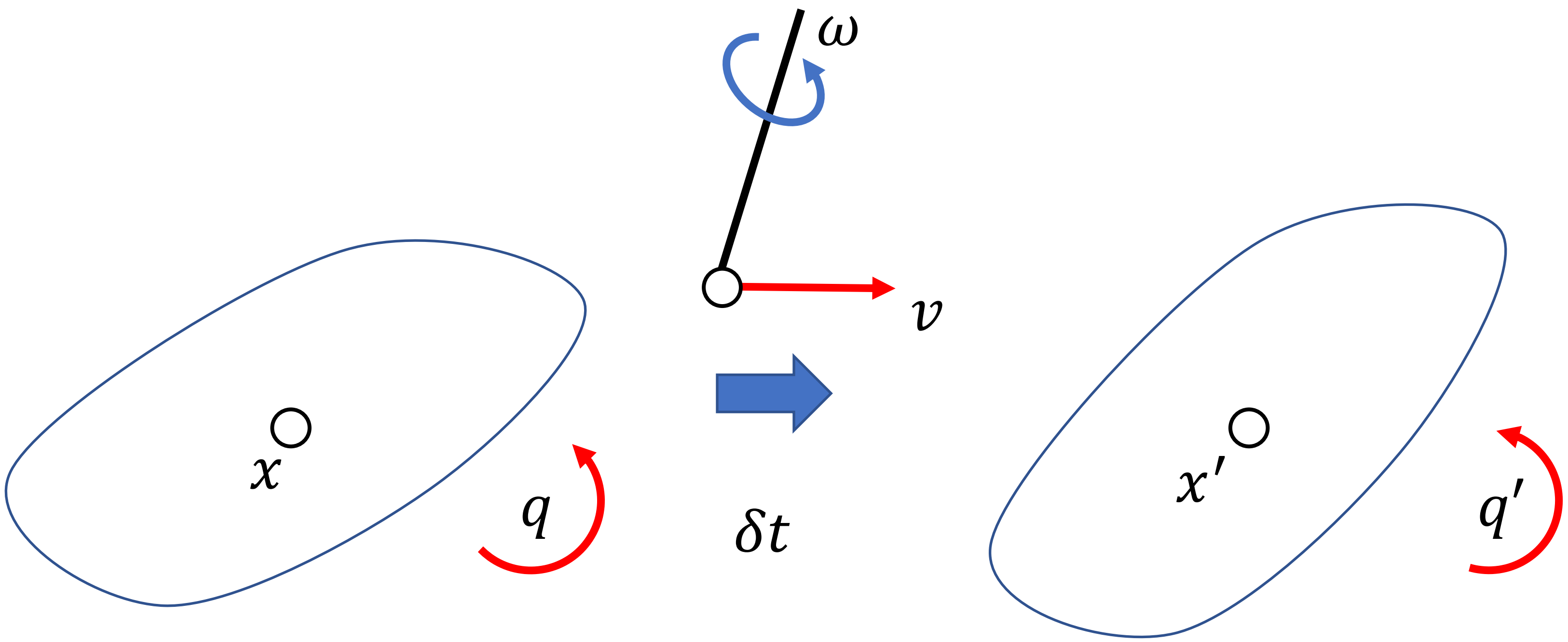


$$\dot{x} = v \quad \Rightarrow \quad x' = x + \delta t \cdot v$$

$$\dot{R} = [\omega]_\times R \quad \Rightarrow \quad R' = R + \delta t \cdot [\omega]_\times R$$

Need orthogonalization!

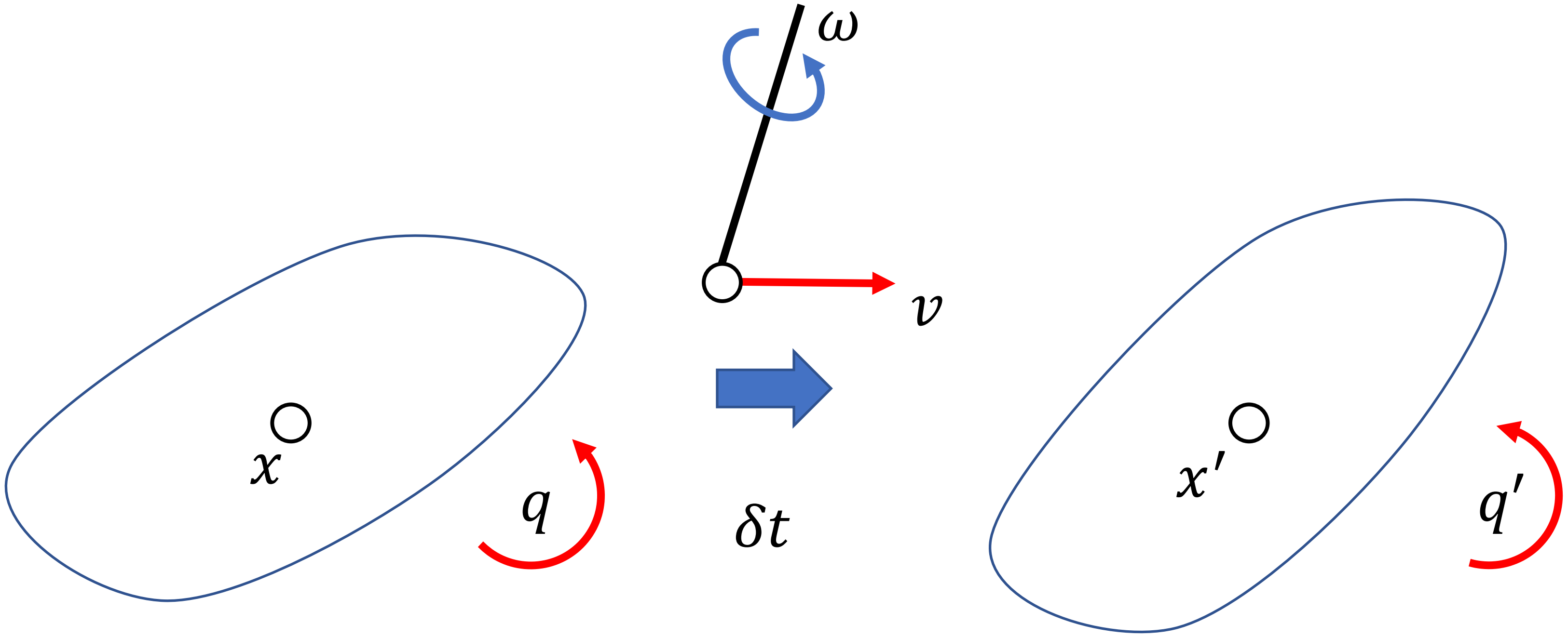
Numerical Integration: Quaternion



$$\dot{x} = v \qquad \Rightarrow \qquad x' = x + \delta t \cdot v$$

$$\dot{q} = ? \qquad \qquad q' = q + \delta t \cdot \dot{q}$$

Numerical Integration: Quaternion



$$\dot{x} = v \quad \Rightarrow \quad x' = x + \delta t \cdot v$$

$$\dot{q} = \frac{1}{2}\bar{\omega}q \quad \Rightarrow \quad q' = q + \delta t \cdot \dot{q}$$

$\bar{\omega} = (0, \omega)$

Need Normalization!

Kinematics vs. Dynamics

Kinematics

x, R

v, ω

a, α

$\ddot{x}, \ddot{\omega}$

…

m, I

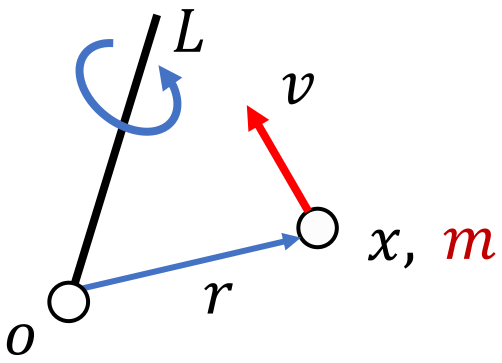
⟺

Dynamics

p, L

F, τ

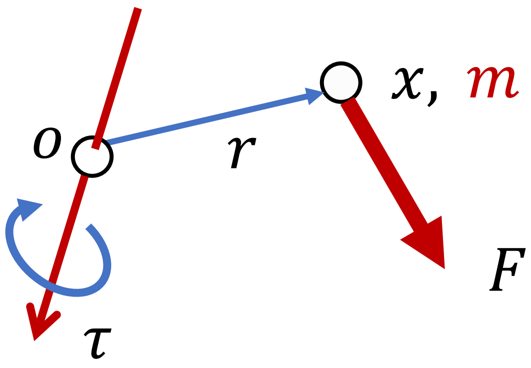
Linear and Angular Momentum of a Particle



$p = m\,v$ — Linear momentum of x

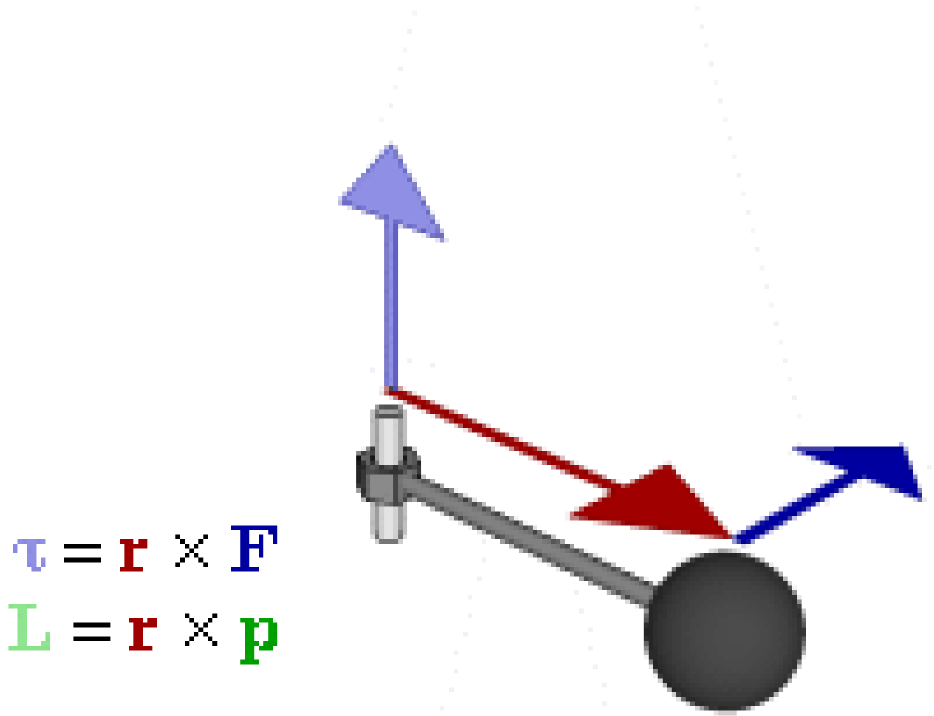
$L = m\,r \times v$ — Angular momentum of x w.r.t. o

Force and Torque



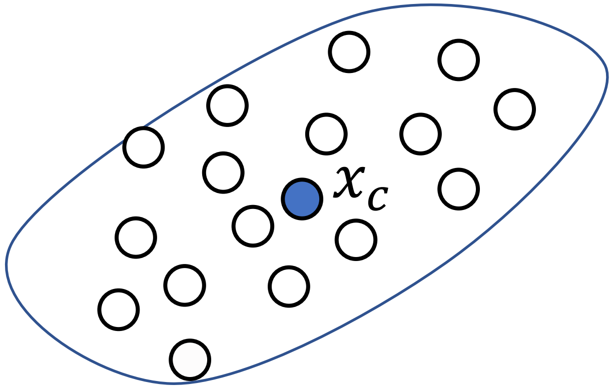
$$\tau = r \times F$$

Torque and Angular Momentum



https://en.wikipedia.org/wiki/Torque

Rigid Body as a Collection of Particles



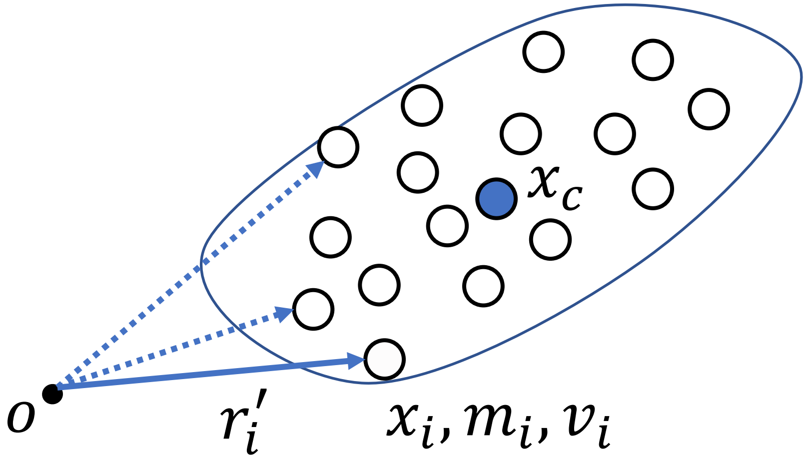
m_i, x_i, v_i

$$M = \sum_i m_i$$

$$x_c = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

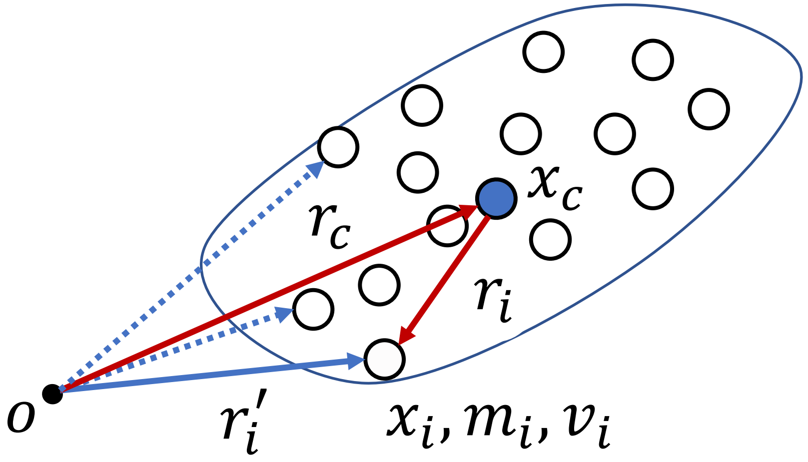
$$v_c = \frac{\sum_i m_i v_i}{\sum_i m_i}$$

Moments of a Rigid Body



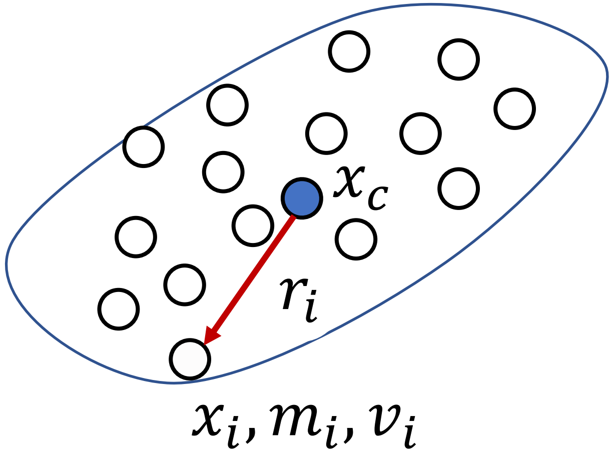
$$p = \sum_i m_i v_i \qquad L_o = \sum_i m_i r_i' \times v_i$$

Angular Moment of a Rigid Body



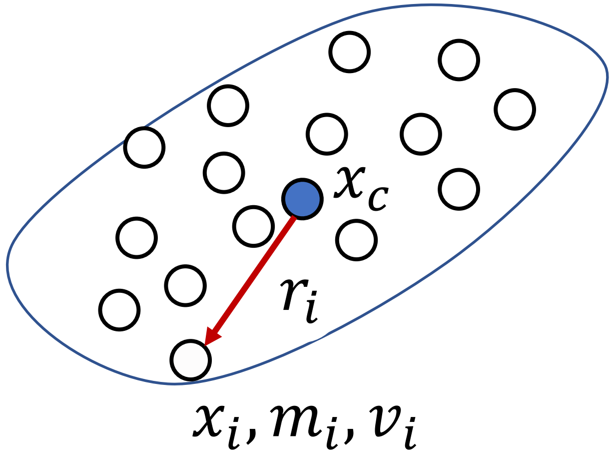
$$L_o = \sum_i m_i r_i' \times v_i = M r_c \times v_c + \sum_i m_i r_i \times v_i$$

Angular Moment of a Rigid Body



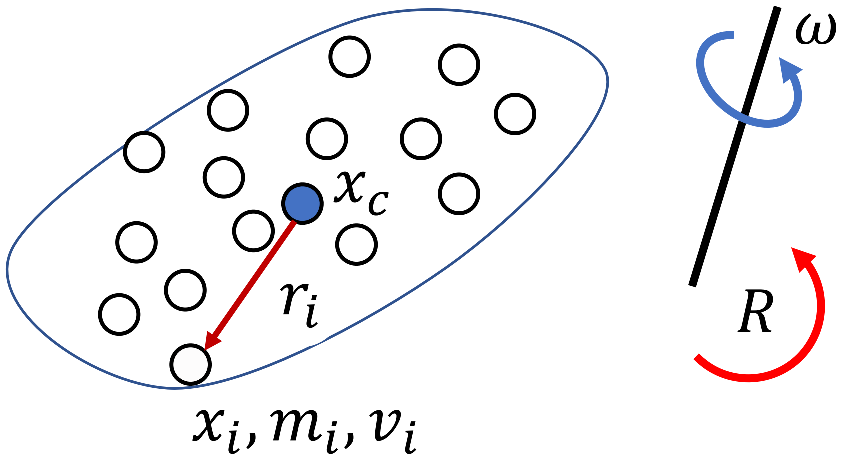
$$L_{x_c} = \sum_i m_i r_i \times v_i$$

Angular Moment of a Rigid Body



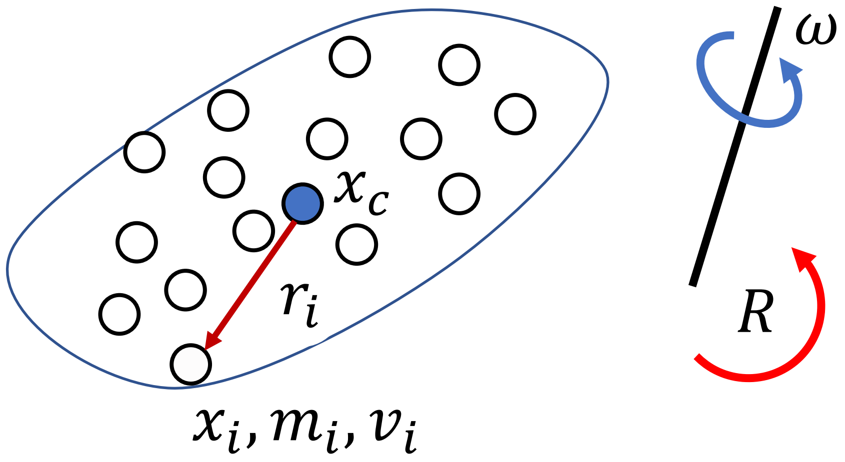
$$L = \sum_i m_i r_i \times v_i$$

Angular Moment of a Rigid Body



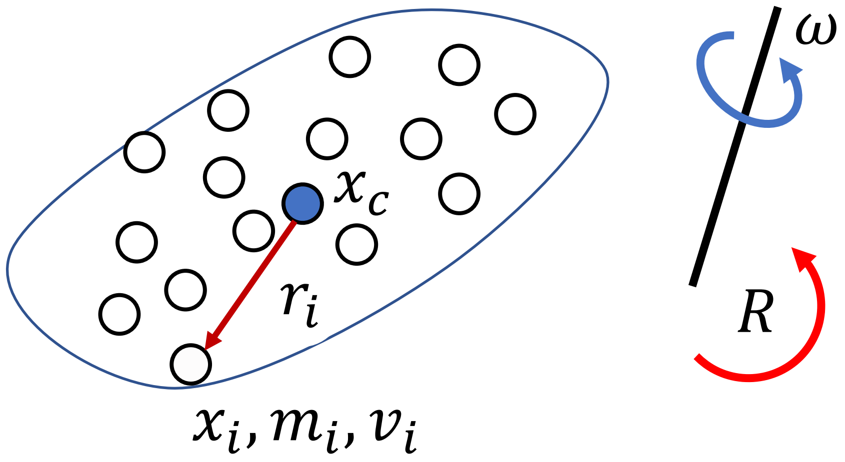
$$L = \sum_i m_i r_i \times v_i$$

Angular Moment of a Rigid Body



$$L = \sum_i m_i r_i \times (\omega \times r_i)$$

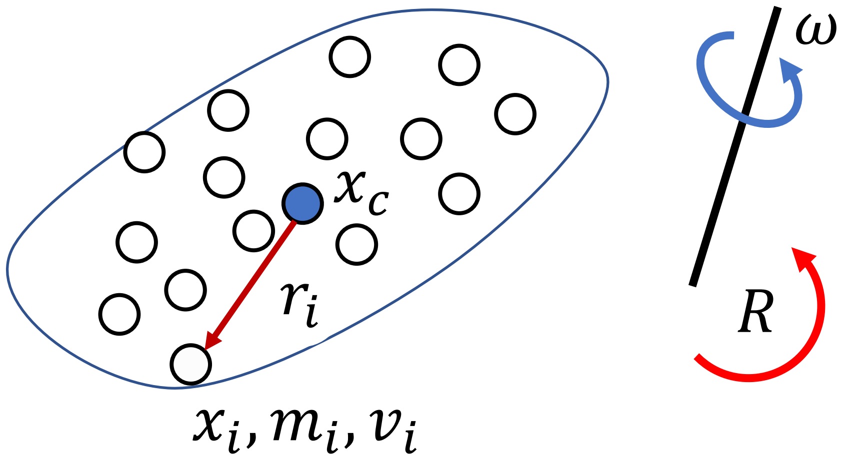
Angular Moment of a Rigid Body



$$L = \sum_i -m_i [r_i]_\times^2 \, \omega$$

$$[a]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

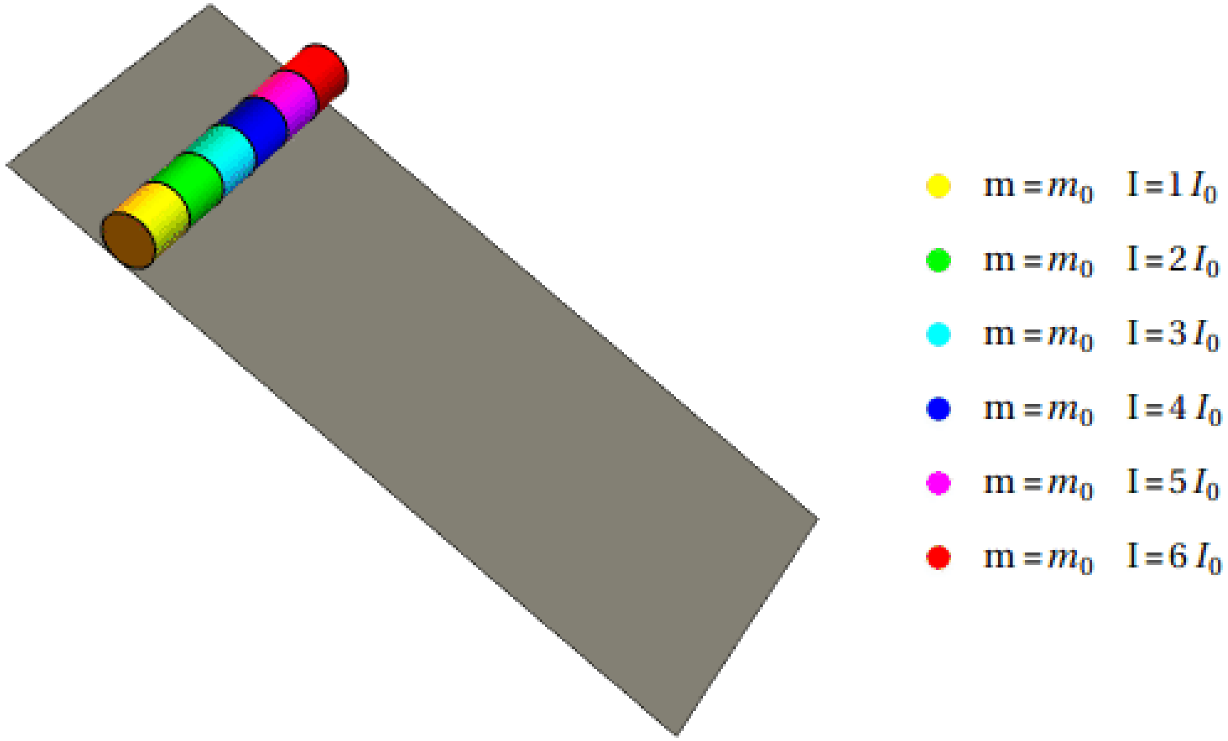
Angular Moment of a Rigid Body



$$L = I\omega$$

Moment of Inertia: $I = \sum_i -m_i [r_i]_\times^2$

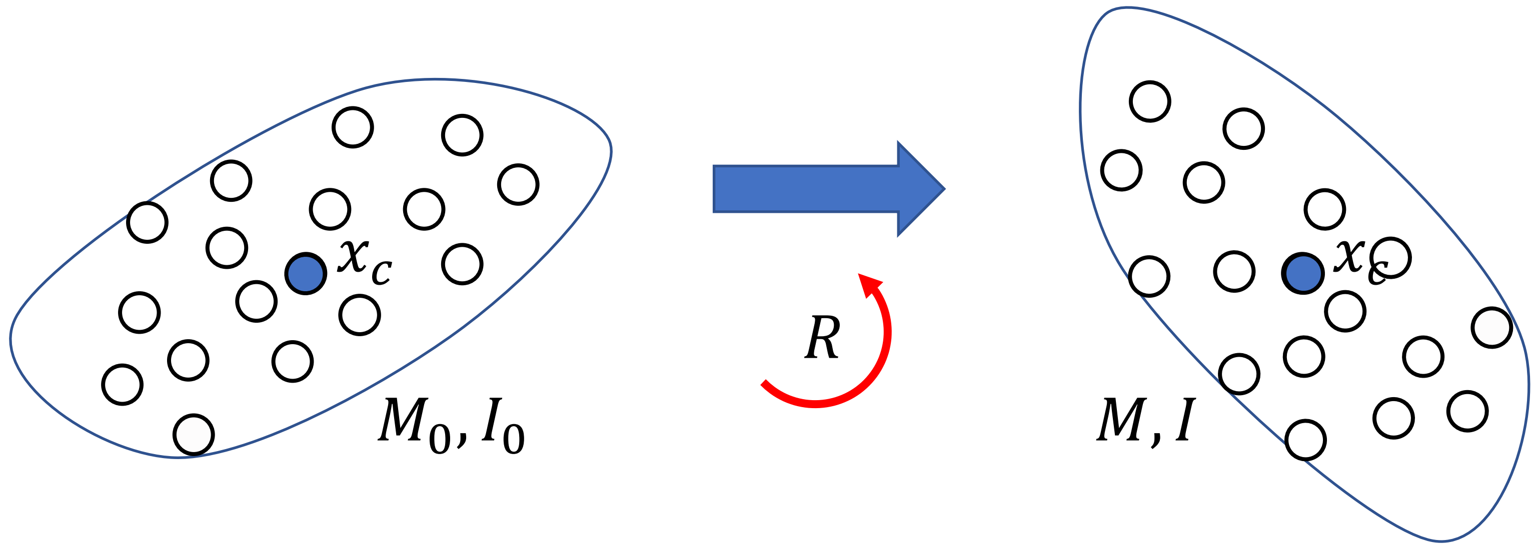
Moment of Inertia



Moment of Inertia

https://en.wikipedia.org/wiki/Moment_of_inertia

Rotation of Moment of Inertia



$$M = M_0$$

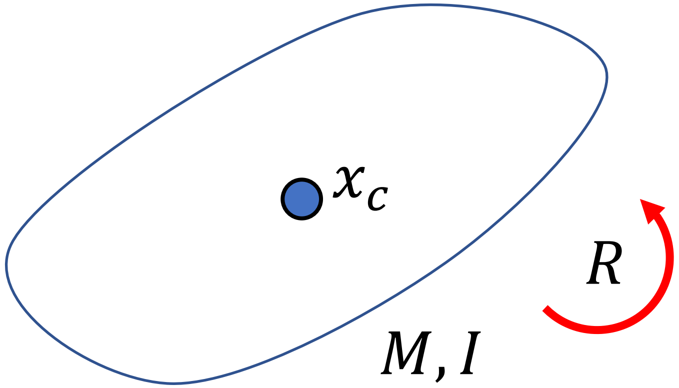
$$I = R I_0 R^T$$

$$(Rr) \times x = R\left(r \times \left(R^T x\right)\right)$$

$$[Rr]_\times = R[r]_\times R^T$$

$$[Rr]_\times^2 = R[r]_\times^2 R^T$$

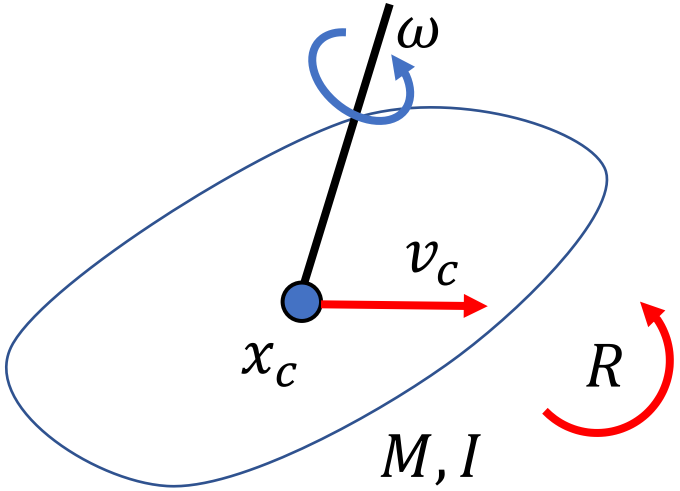
Principal Axes of Moment of Inertia



Eigendecomposition ⇒ $I = RI_0R^T$

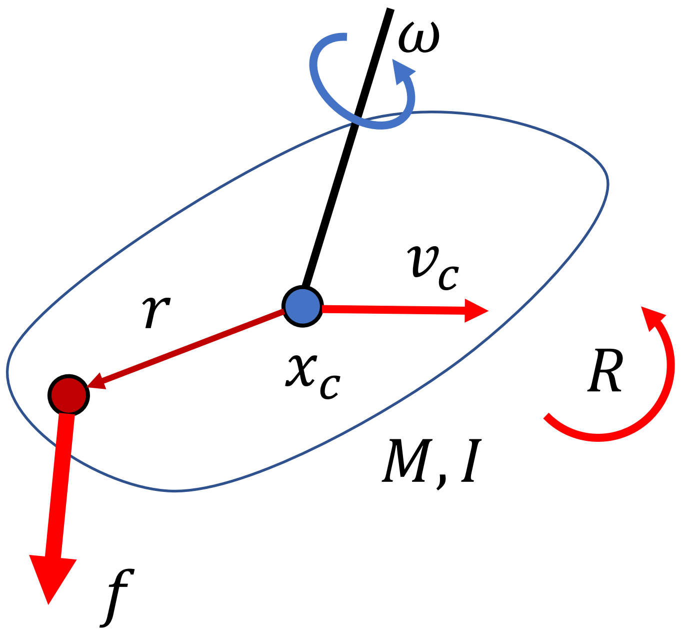
$$I_0 = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \mathrm{diag}(I_1, I_2, I_3)$$

Center of Momentum (CoM) Frame

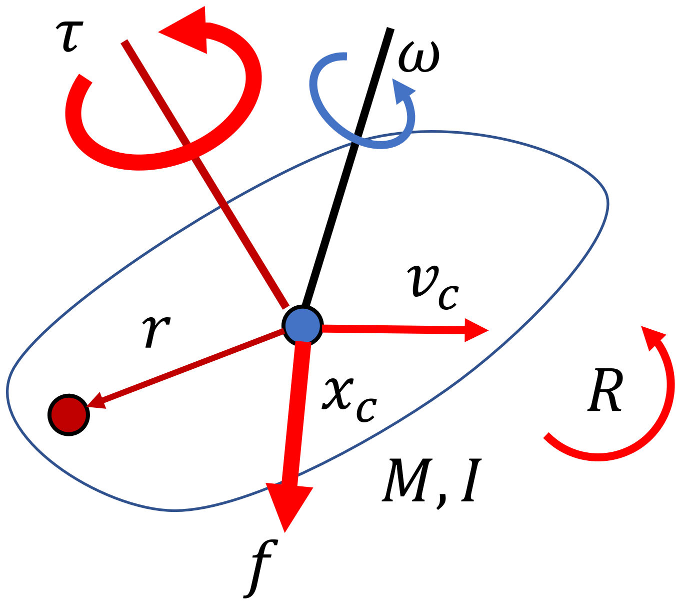


$$p = Mv_c \qquad\qquad L = I\omega$$

Force on a Rigid Body

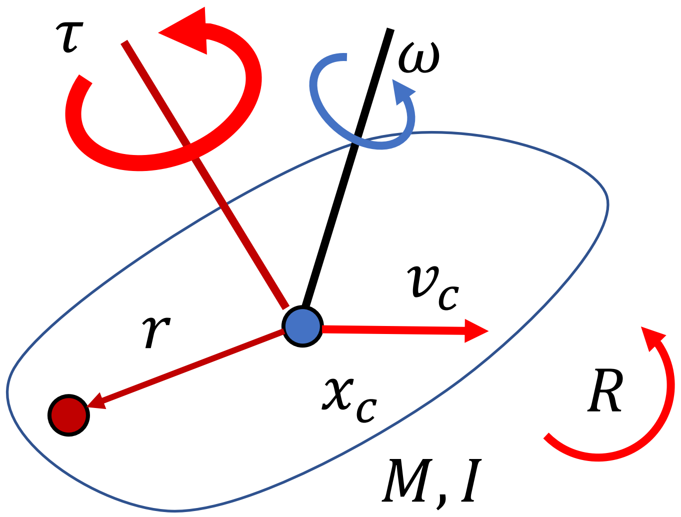


Force on a Rigid Body



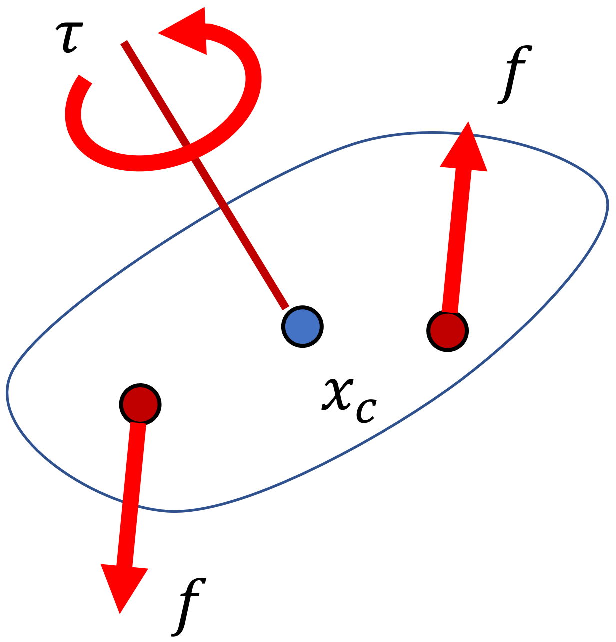
$$\tau = r \times f$$

Torque on a Rigid Body



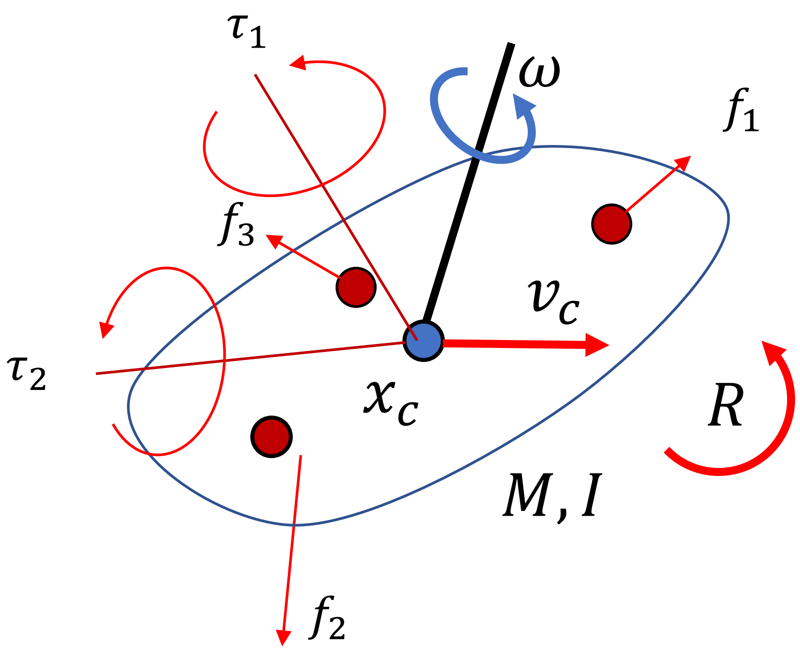
$$\tau = ???$$

Parallel Forces and Torques



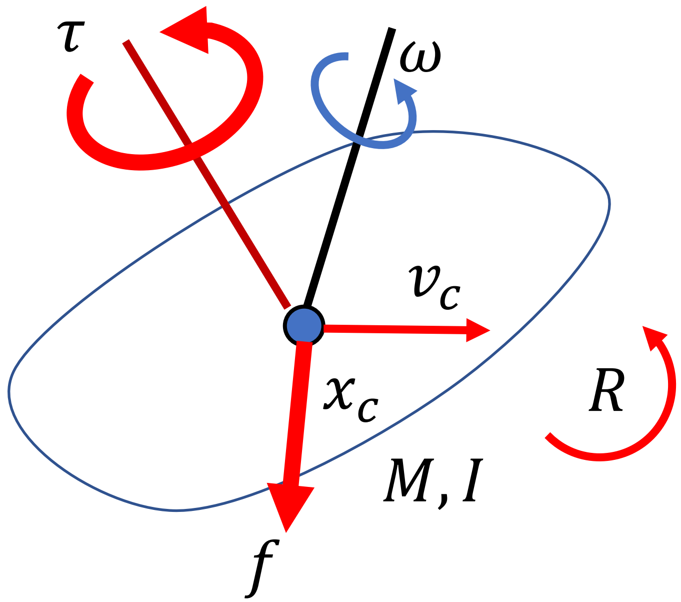
$$\tau = ???$$

Center of Momentum (CoM) Frame



$p = Mv_c$ $\quad\quad$ $L = I\omega$

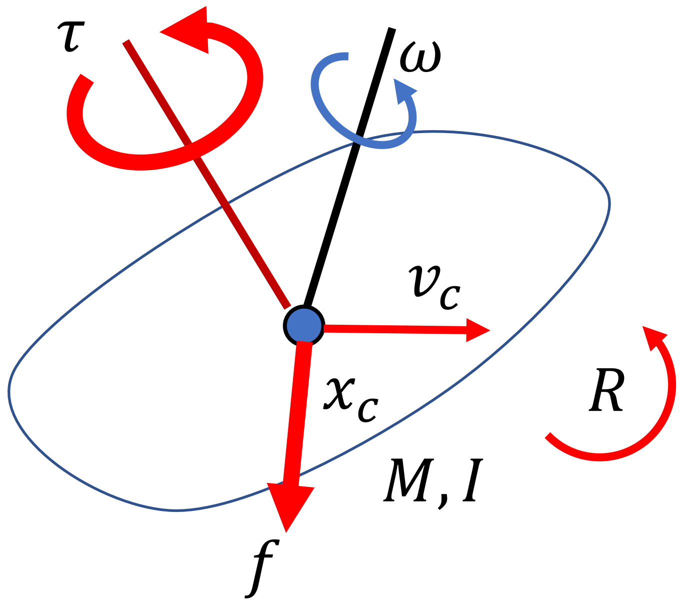
Center of Momentum (CoM) Frame



$p = Mv_c$ $\qquad$ $L = I\omega$

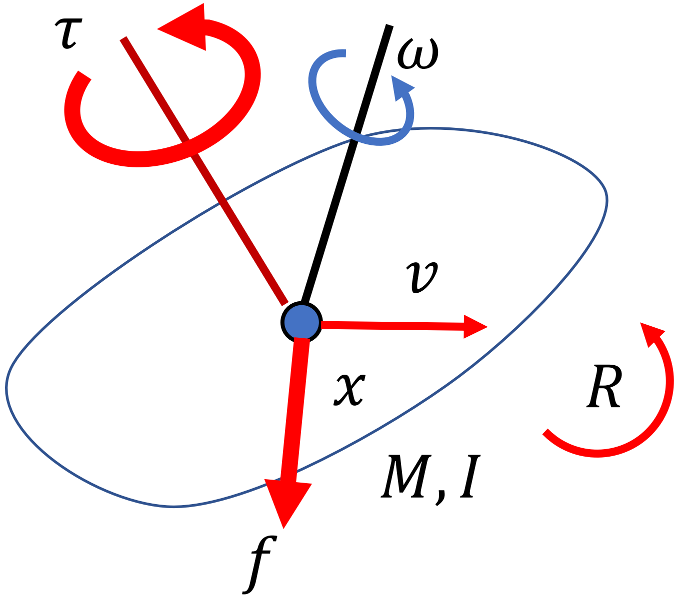
$f = \sum_i f_i$ $\qquad$ $\tau = \sum_i \tau_i$

Equation of Motion of Rigid Body



Kinematics		Dynamics
	m, I	
x, R	⟺	p, L
v, ω		f, τ

Equation of Motion of Rigid Body



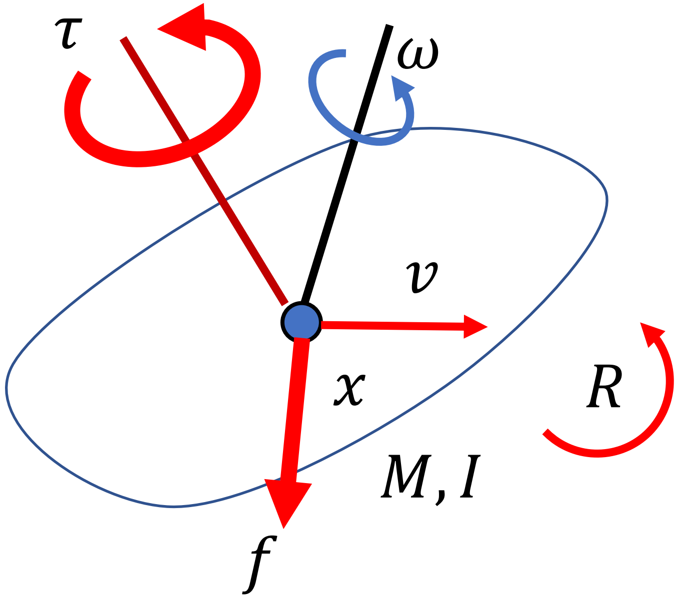
x, R, v, ω

$p = Mv$

$L = I\omega$

Newton's Second Law: $f = Ma$

Equation of Motion of Rigid Body



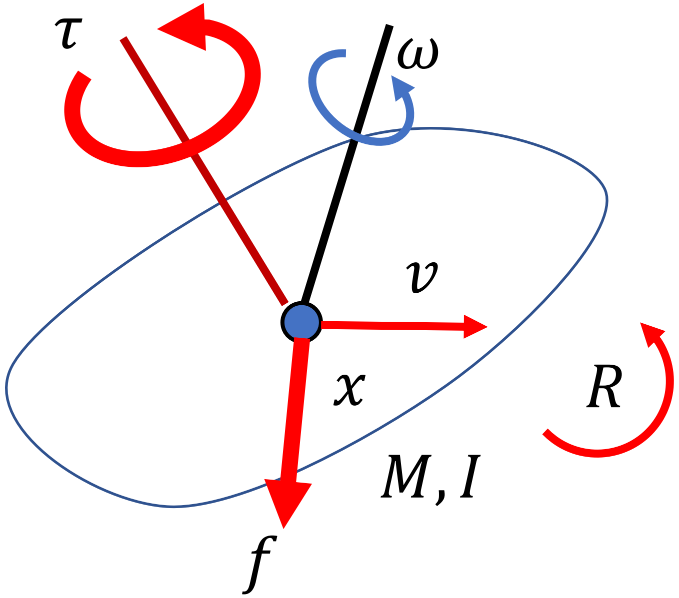
$$x, R, v, \omega$$

$$p = Mv$$

$$L = I\omega$$

Newton's Second Law: $\frac{dp}{dt} = f$

Equation of Motion of Rigid Body



x, R, v, ω

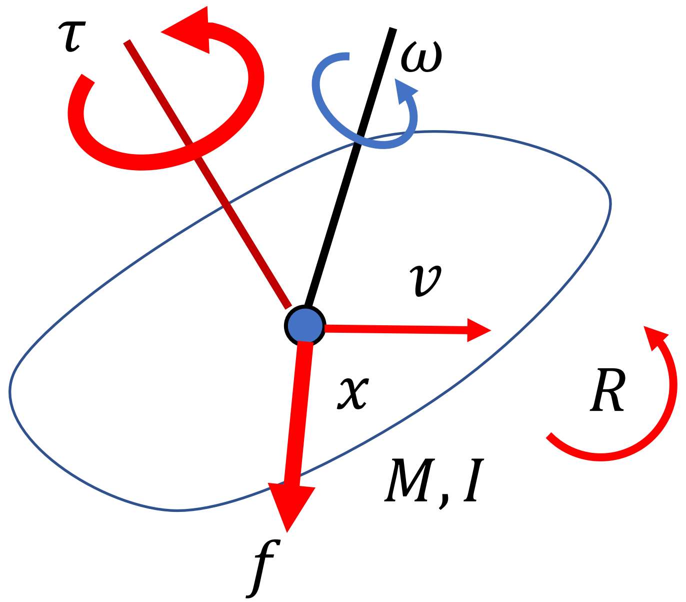
$p = Mv$

$L = I\omega$

Newton's Second Law: $\frac{dp}{dt} = f$

Euler's laws of motion: $\frac{dL}{dt} = \tau$

Equation of Motion of Rigid Body



x, R, v, ω

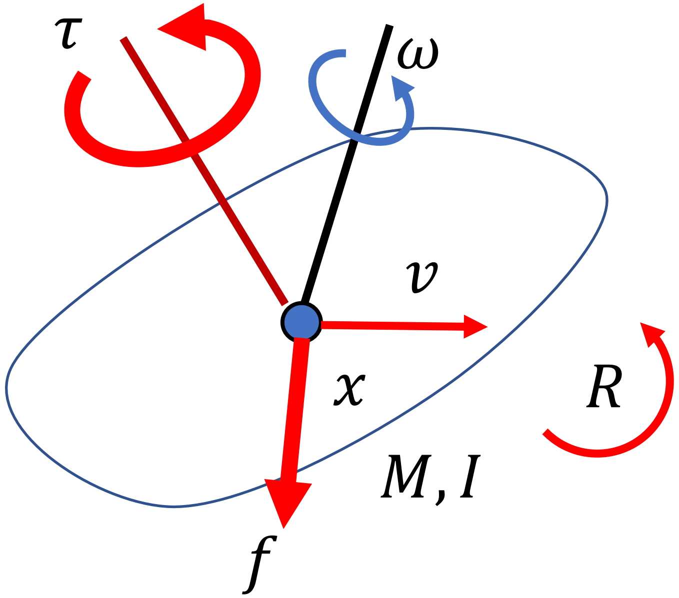
$p = Mv$

$L = I\omega$

Newton's Second Law: $\frac{dp}{dt} = f$ ➡ $M\dot{v} = f$

Euler's laws of motion: $\frac{dL}{dt} = \tau$ ➡ $I\dot{\omega} + \dot{I}\omega = \tau$

Equation of Motion of Rigid Body



x, R, v, ω

$p = Mv$

$L = I\omega$

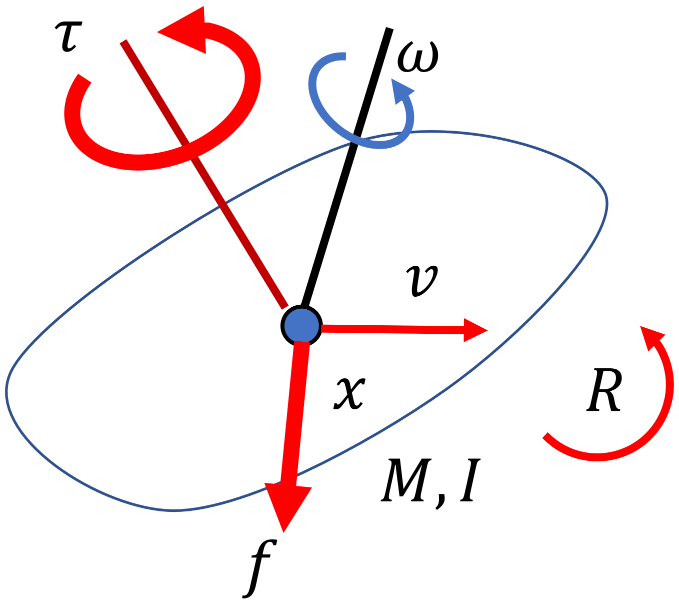
$$\dot{I} = \frac{d}{dt}\left(RI_0R^T\right)$$
$$= \dot{R}I_0R^T + RI_0\dot{R}^T$$
$$= [\omega]_\times RI_0R^T + RI_0R^T[\omega]_\times^T$$

$$\dot{I}\omega = \omega \times I\omega + I(-\omega \times \omega)$$

Newton's Second Law: $\frac{dp}{dt} = f \Rightarrow M\dot{v} = f$

Euler's laws of motion: $\frac{dL}{dt} = \tau \Rightarrow I\dot{\omega} + \omega \times I\omega = \tau$

Newton–Euler Equations



x, R, v, ω

$p = mv$

$L = I\omega$

$$\begin{bmatrix} m\mathrm{I}_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega \times I\omega \end{bmatrix} = \begin{bmatrix} f \\ \tau \end{bmatrix}$$

Numerical Integration

$$\begin{bmatrix} m\mathrm{I}_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega \times I\omega \end{bmatrix} = \begin{bmatrix} f \\ \tau \end{bmatrix}$$

⬇

$$\frac{1}{h} \begin{bmatrix} m\mathrm{I}_3 & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} v_{n+1} - v_n \\ \omega_{n+1} - \omega_n \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n \times I_n \omega_n \end{bmatrix} = \begin{bmatrix} f \\ \tau \end{bmatrix}$$

Rigid Body Simulation

$$\frac{1}{h}\begin{bmatrix} m\mathrm{I}_3 & 0 \\ 0 & I_n \end{bmatrix}\begin{bmatrix} v_{n+1} - v_n \\ \omega_{n+1} - \omega_n \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n \times I_n \omega_n \end{bmatrix} = \begin{bmatrix} f \\ \tau \end{bmatrix}$$

$$I_n = R_n I_0 R_n^T$$

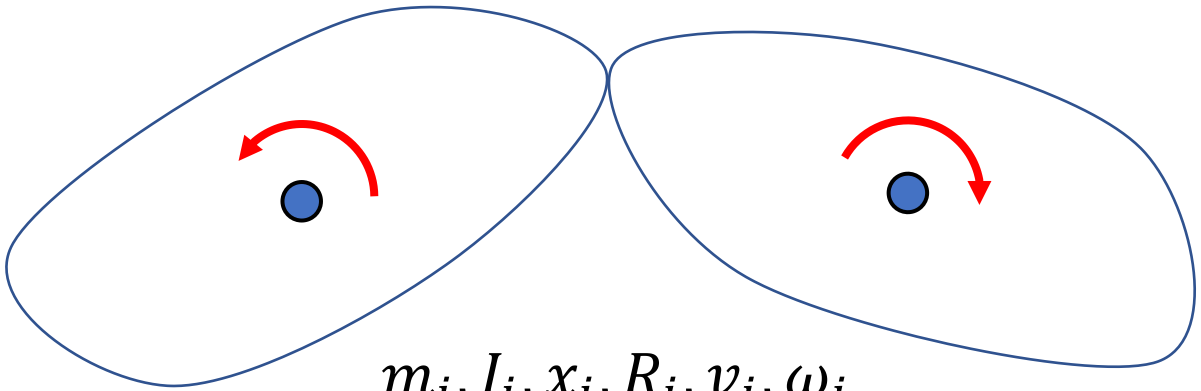
$$v_{n+1} = \cdots$$

$$\omega_{n+1} = \cdots$$

$$x_{n+1} = x_n + h v_{n+1}$$

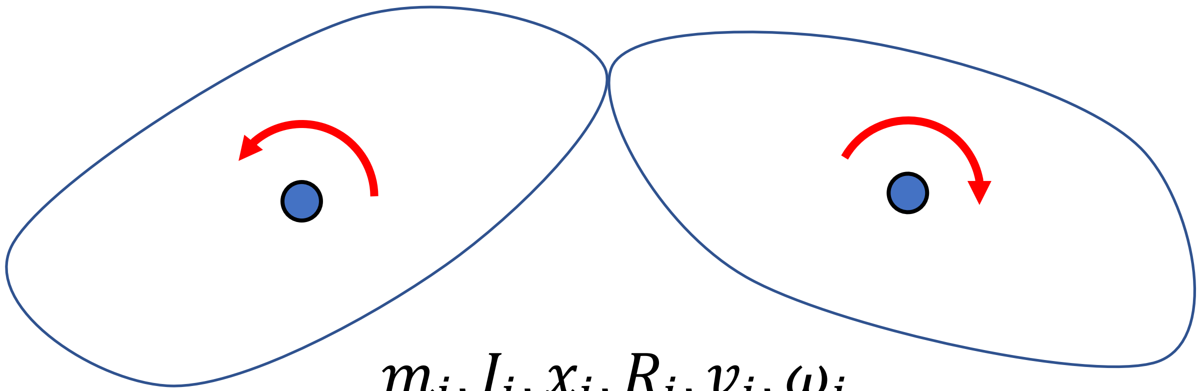
$$q_{n+1} = q_n + \frac{1}{2} h \bar{\omega}_{n+1} q$$

A System with Two Links



$m_i, I_i, x_i, R_i, v_i, \omega_i$

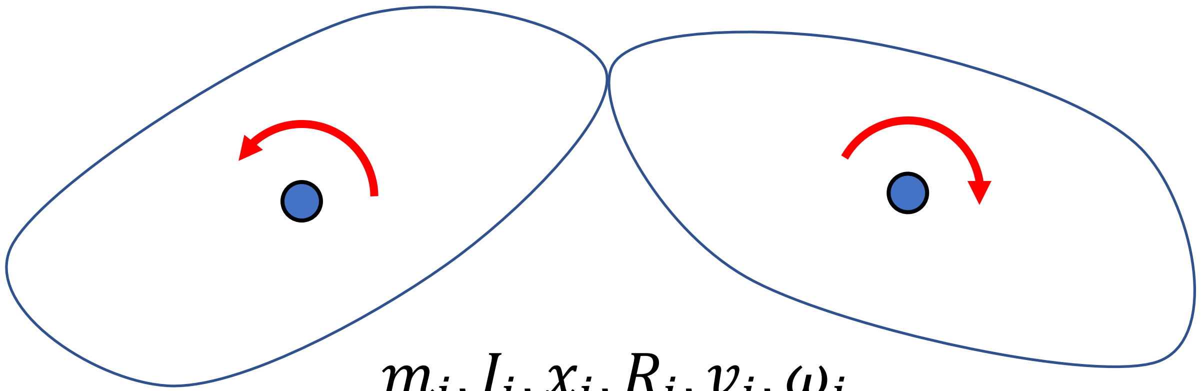
A System with Two Links



$$m_i, I_i, x_i, R_i, v_i, \omega_i$$

$$\begin{bmatrix} m_1\mathbf{I}_3 & & & \\ & I_1 & & \\ & & m_2\mathbf{I}_3 & \\ & & & I_2 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{\omega}_1 \\ \dot{v}_2 \\ \dot{\omega}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_1 \times I_1\omega_1 \\ 0 \\ \omega_2 \times I_2\omega_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ \tau_1 \\ f_2 \\ \tau_2 \end{bmatrix}$$

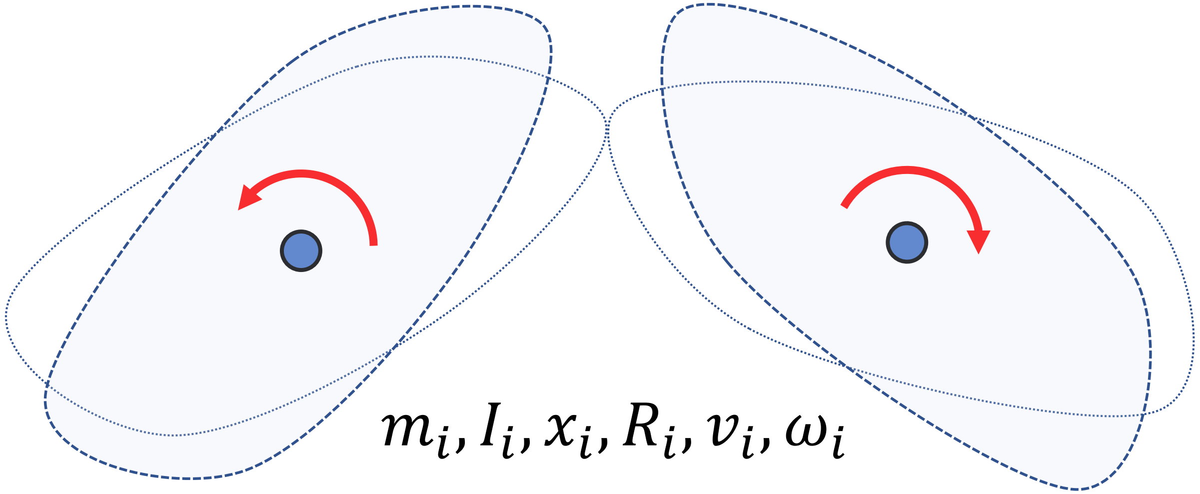
A System with Two Links



$$m_i, I_i, x_i, R_i, v_i, \omega_i$$

$$M\dot{\boldsymbol{v}} + C(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{f}$$

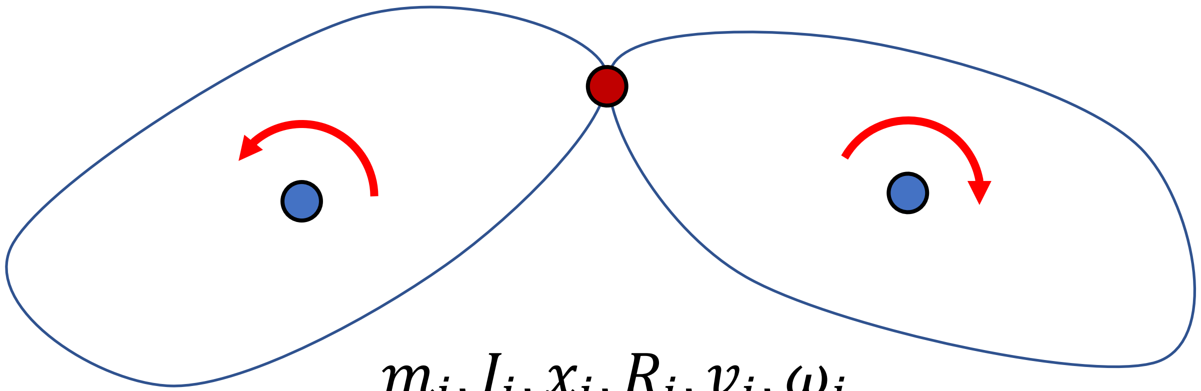
A System with Two Links



$m_i, I_i, x_i, R_i, v_i, \omega_i$

$$M\dot{\boldsymbol{v}} + C(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{f}$$

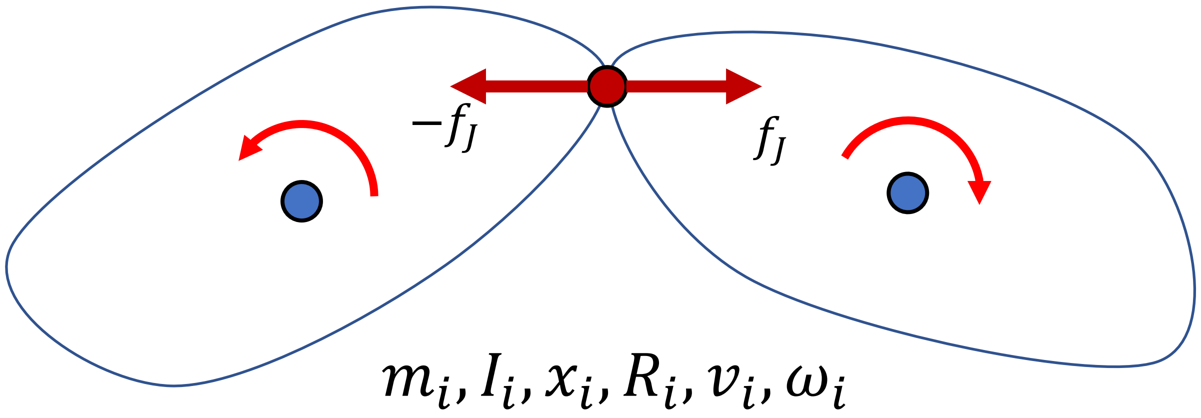
A System with Two Links and a Joint



$m_i, I_i, x_i, R_i, v_i, \omega_i$

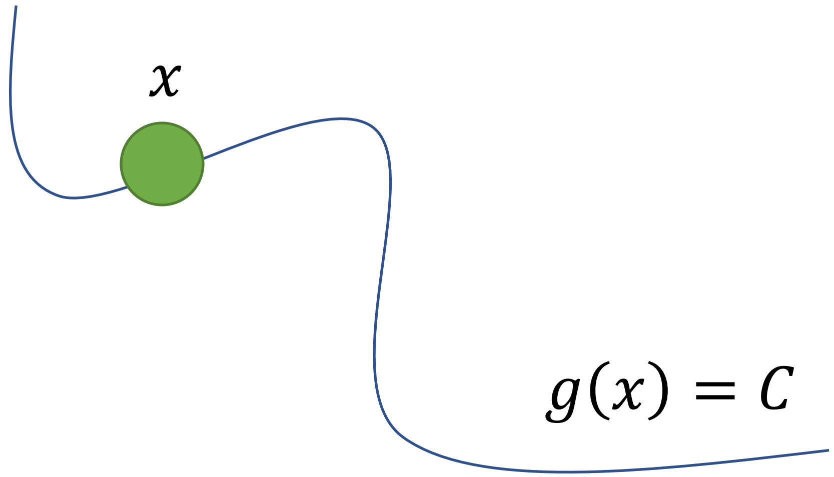
$$M\dot{\boldsymbol{v}} + C(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{f}$$

A System with Two Links and a Joint



$$M\dot{\boldsymbol{v}} + C(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{f} + \boldsymbol{f}_J$$

Constraints

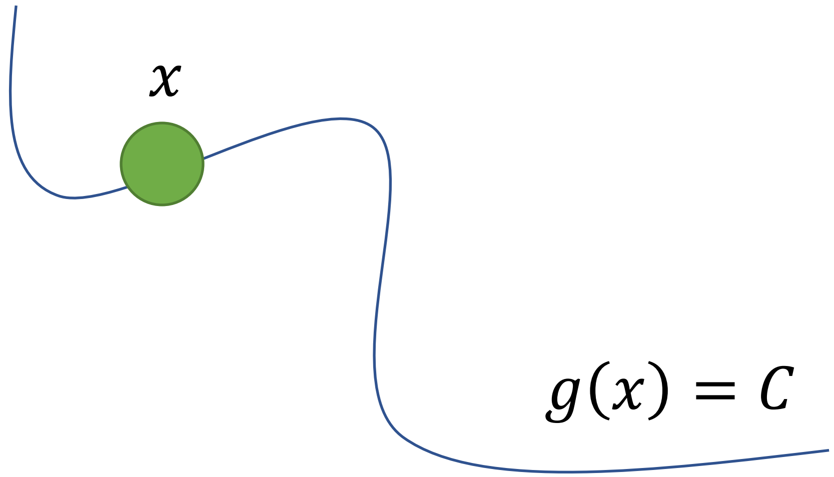


$$g(\boldsymbol{x}) = C$$

$$\Downarrow$$

$$\frac{d}{dt} g(\boldsymbol{x}) = 0$$

Constraints



$$g(\boldsymbol{x}) = C$$

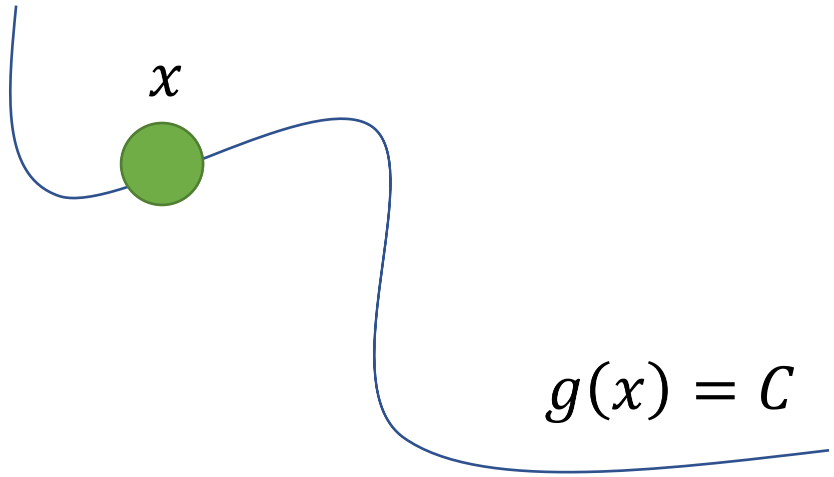
$$\Downarrow$$

$$\frac{d}{dt} g(\boldsymbol{x}) = 0$$

$$\Downarrow$$

$$\frac{\partial g}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt} = 0$$

Constraints



$$g(\boldsymbol{x}) = C$$

$$\Downarrow$$

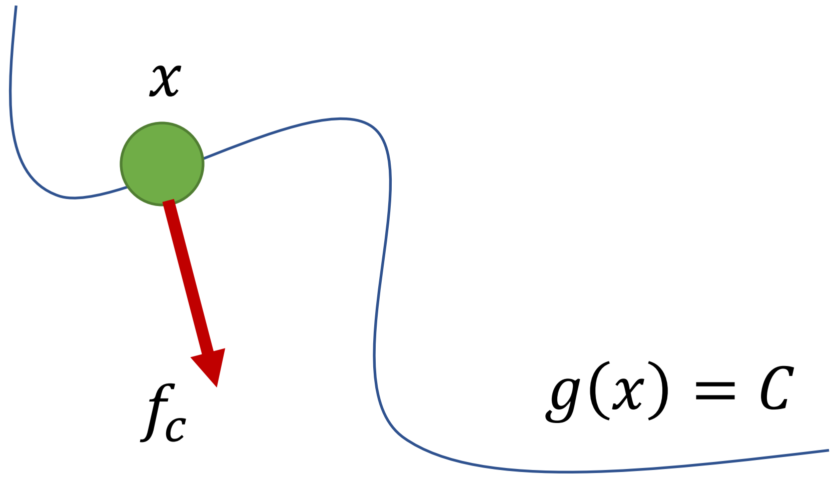
$$\frac{d}{dt} g(\boldsymbol{x}) = 0$$

$$\Downarrow$$

$$J\boldsymbol{v} = 0$$

$$J = [\nabla g]^T$$

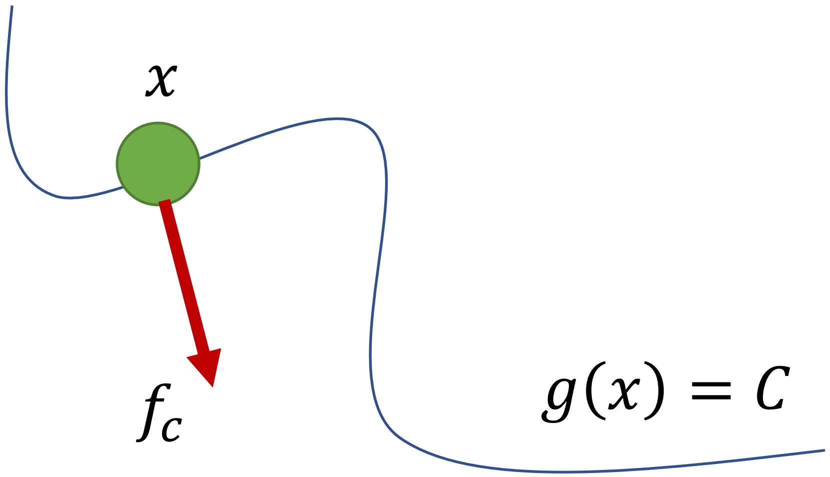
Constraint Force



* Constraint is passive
No energy gain or loss!!!

$$f_c \cdot v = 0$$

Constraint Force



* Constraint is passive
No energy gain or loss!!!

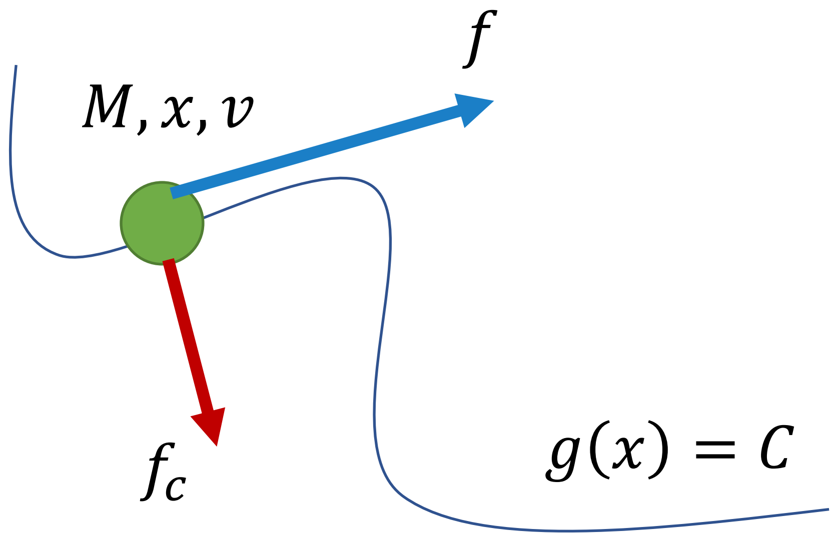
$$f_c \cdot v = 0 \Leftrightarrow f_c^T v = 0$$

$$Jv = 0$$

$$f_c = J^T \lambda$$

unknown

Equation of Motion with Constraints



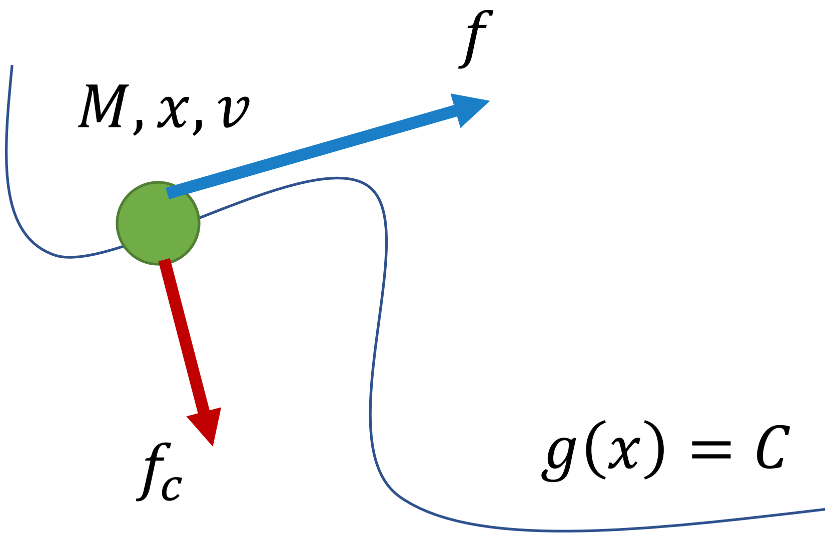
$$M\dot{v} = f + J^T\lambda$$

$$Jv = 0$$

$$M\frac{v_{n+1} - v_n}{h} = f + J^T\lambda$$

$$Jv_{n+1} = 0$$

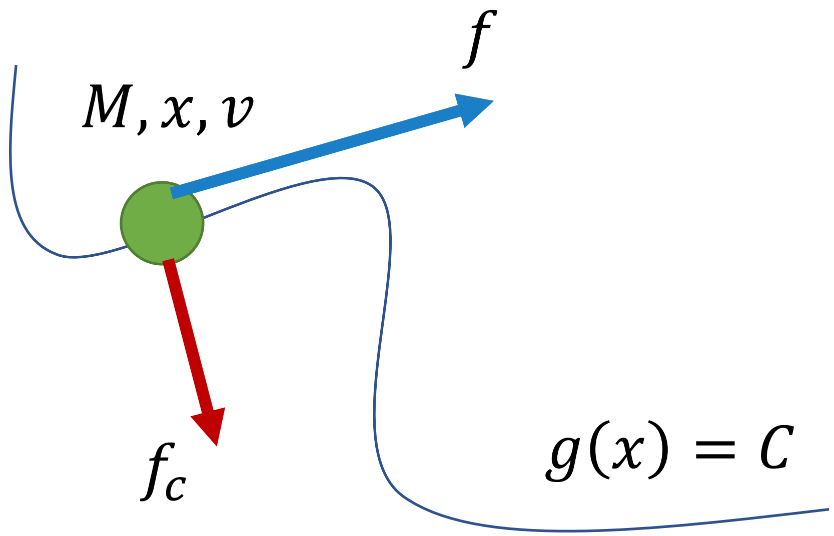
Numerical Solution



$$M \frac{v_{n+1} - v_n}{h} = f + J^T \lambda$$

$$J v_{n+1} = 0$$

Numerical Solution



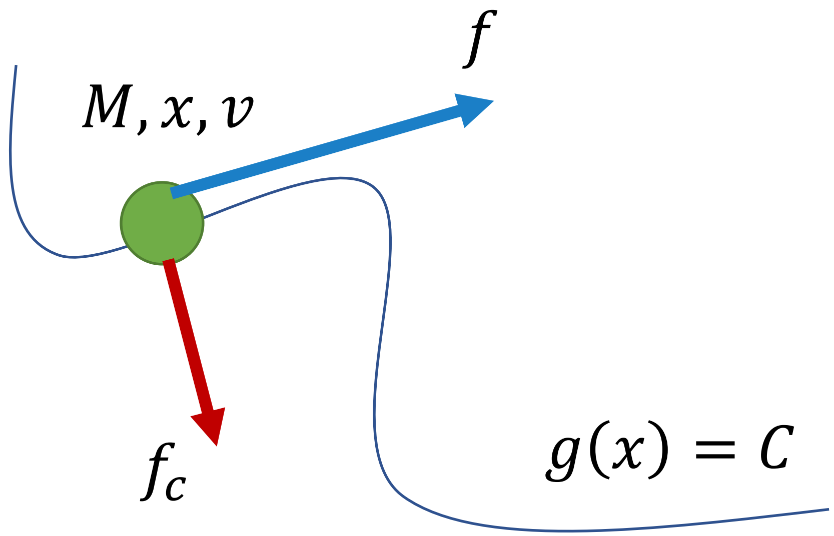
$$M \frac{v_{n+1} - v_n}{h} = f + J^T \lambda$$

$$J v_{n+1} = \mathbf{0}$$

$$J v_{n+1} = \alpha \frac{C - g(x_n)}{h}$$

Correction of numerical errors
α: error reduction parameter (ERP)

Numerical Solution



$$M\frac{v_{n+1} - v_n}{h} = f + J^T\lambda$$

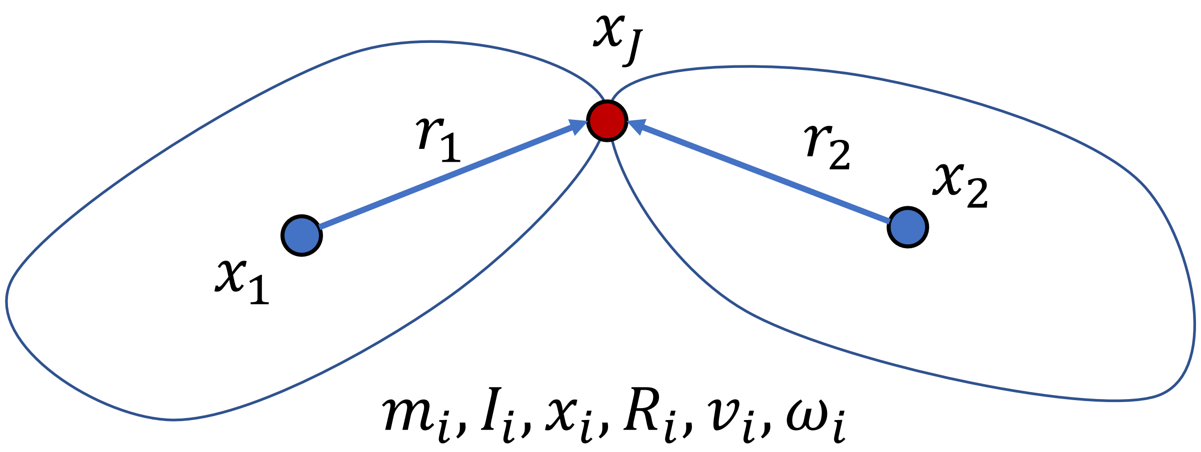
$$Jv_{n+1} = b_n$$

$$JM^{-1}J^T\lambda = c_n$$

$$(JM^{-1}J^T + \beta\mathbf{I})\lambda = c_n$$

β: constraint force mixing (CFM)

Joint Constraint

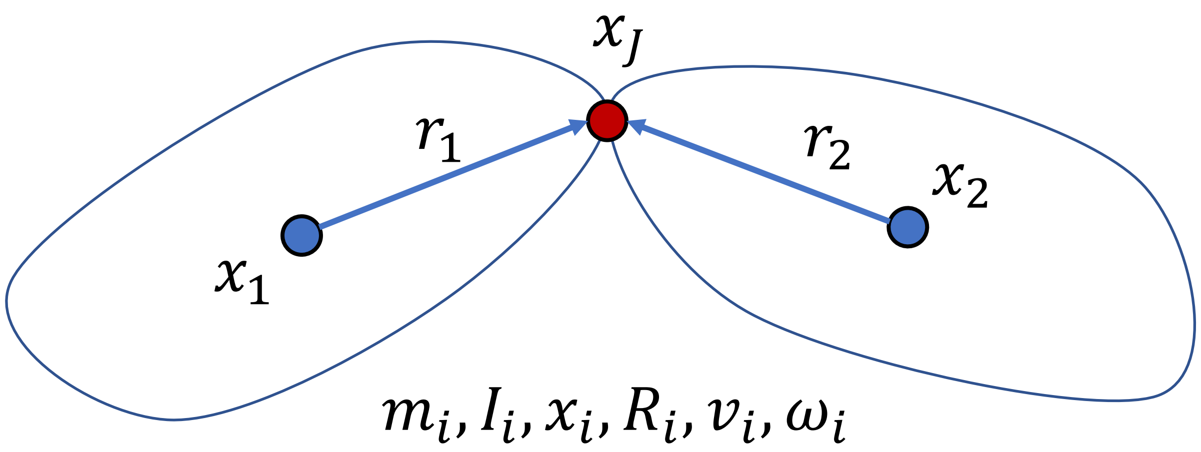


$$x_1 + R_1 r_1 = x_J = x_2 + R_2 r_2$$

d/dt

$$v_1 + \omega_1 \times r_1 = v_2 + \omega_2 \times r_2$$

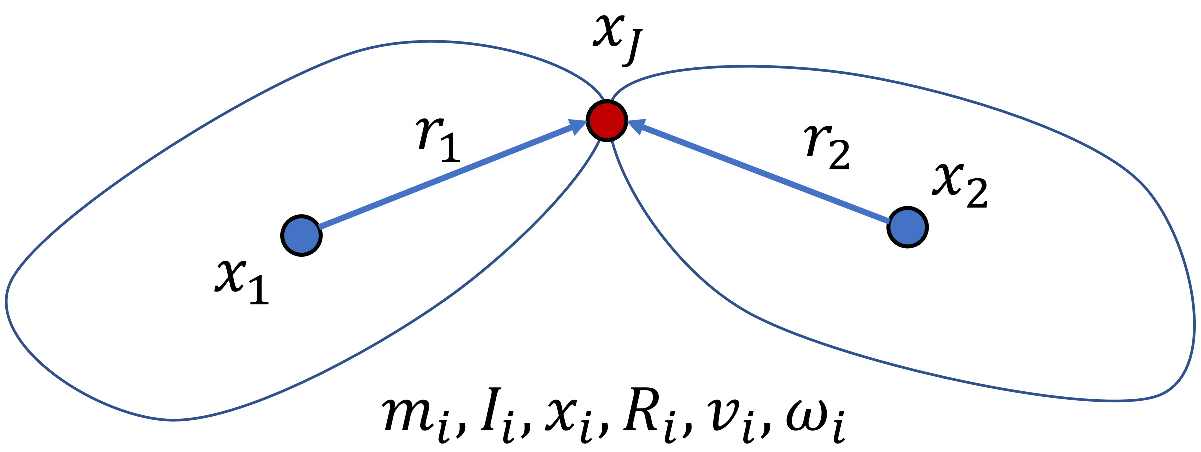
Joint Constraint



$$\begin{bmatrix} I_3 & -[r_1]_\times & -I_3 & [r_2]_\times \end{bmatrix} \begin{bmatrix} v_1 \\ w_1 \\ v_2 \\ w_2 \end{bmatrix} = 0$$

$$Jv = 0$$

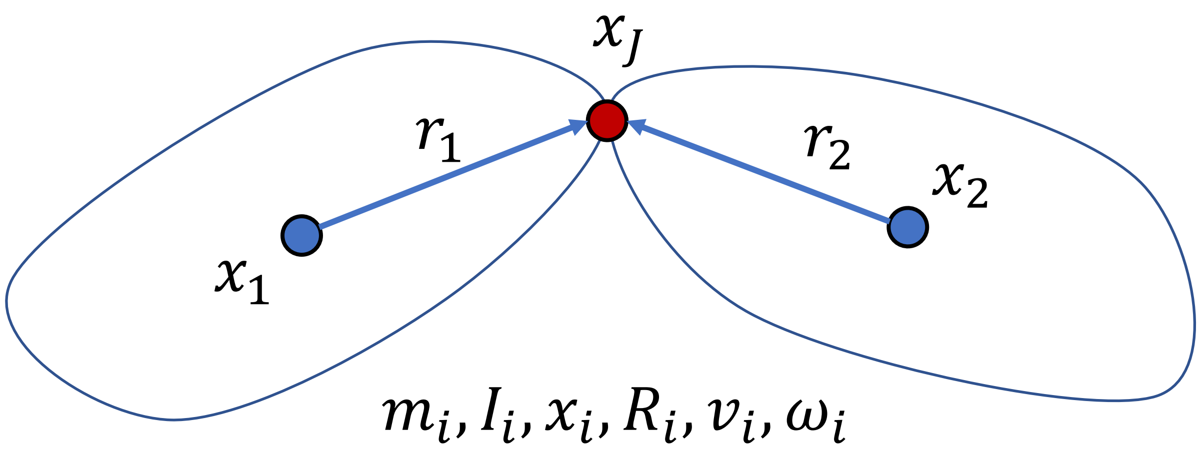
A System with Two Links and a Joint



$$M\dot{\boldsymbol{v}} + C(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{f} + \boldsymbol{J}^T\boldsymbol{\lambda}$$

$$Jv = 0$$

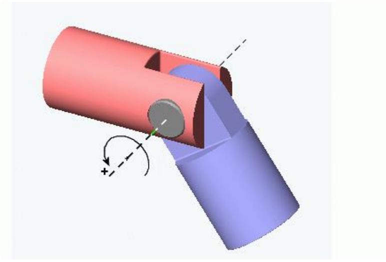
A System with Two Links and a Joint



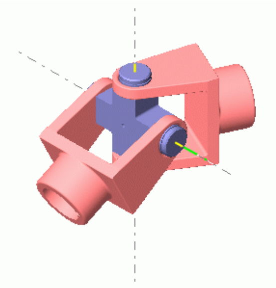
$$\begin{bmatrix} m_1 \mathbf{I}_3 & & & \\ & I_1 & & \\ & & m_2 \mathbf{I}_3 & \\ & & & I_2 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{\omega}_1 \\ \dot{v}_2 \\ \dot{\omega}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_1 \times I_1 \omega_1 \\ 0 \\ \omega_2 \times I_2 \omega_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ \tau_1 \\ f_2 \\ \tau_2 \end{bmatrix} + \begin{bmatrix} I_3 \\ [r_1]_\times \\ -I_3 \\ -[r_2]_\times \end{bmatrix} \lambda$$

$$Jv = 0$$

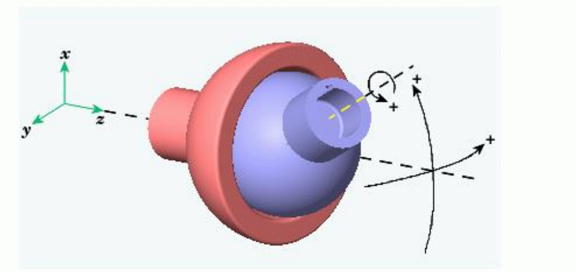
Different Types of Joints



Hinge joint
Revolute joint



Universal joint

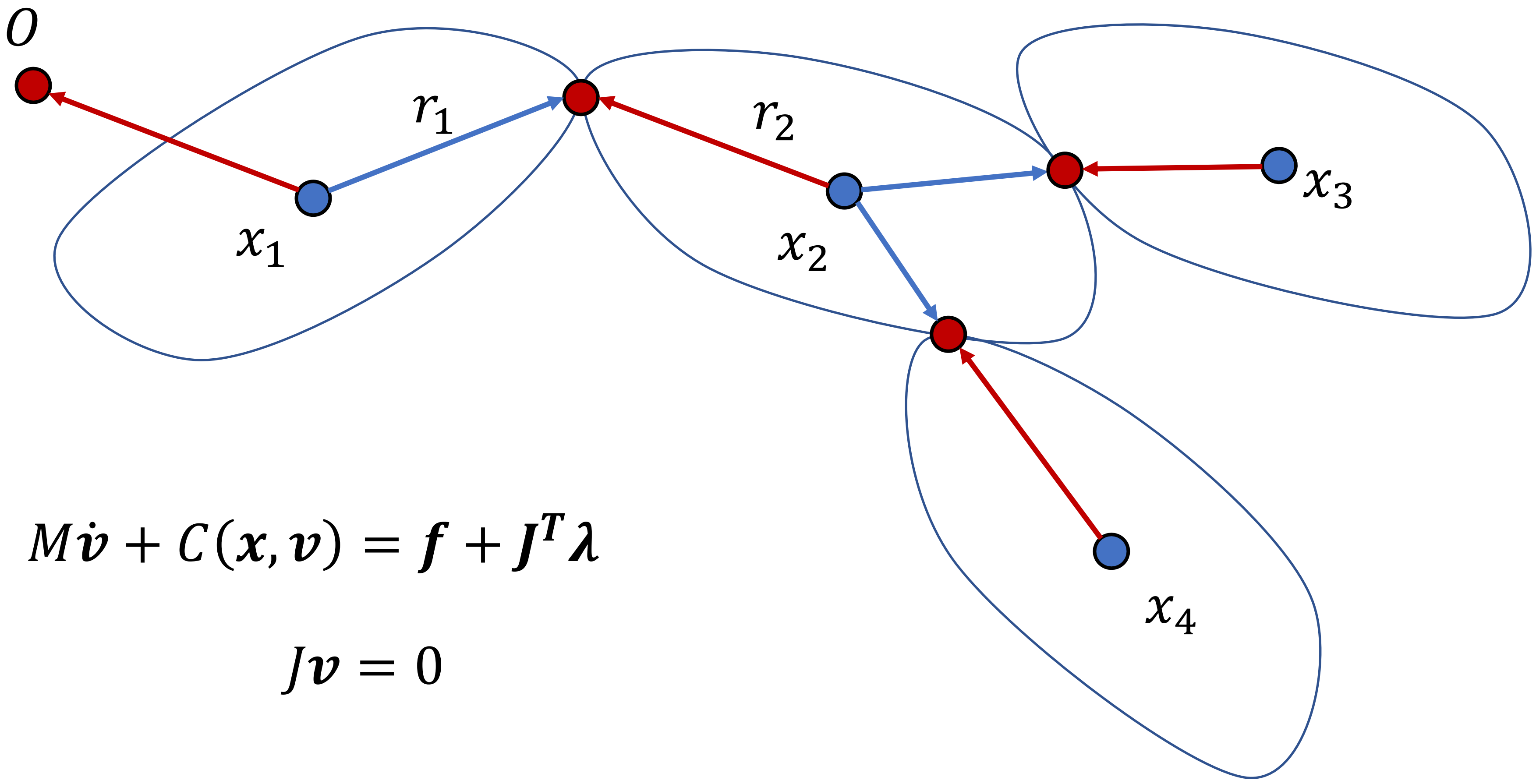


Ball-and-socket

$$\begin{bmatrix} I_3 & -[r_1]_\times & -I_3 & [r_2]_\times \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} v_1 \\ w_1 \\ v_2 \\ w_2 \end{bmatrix} = 0$$

A System with Many Links Joints

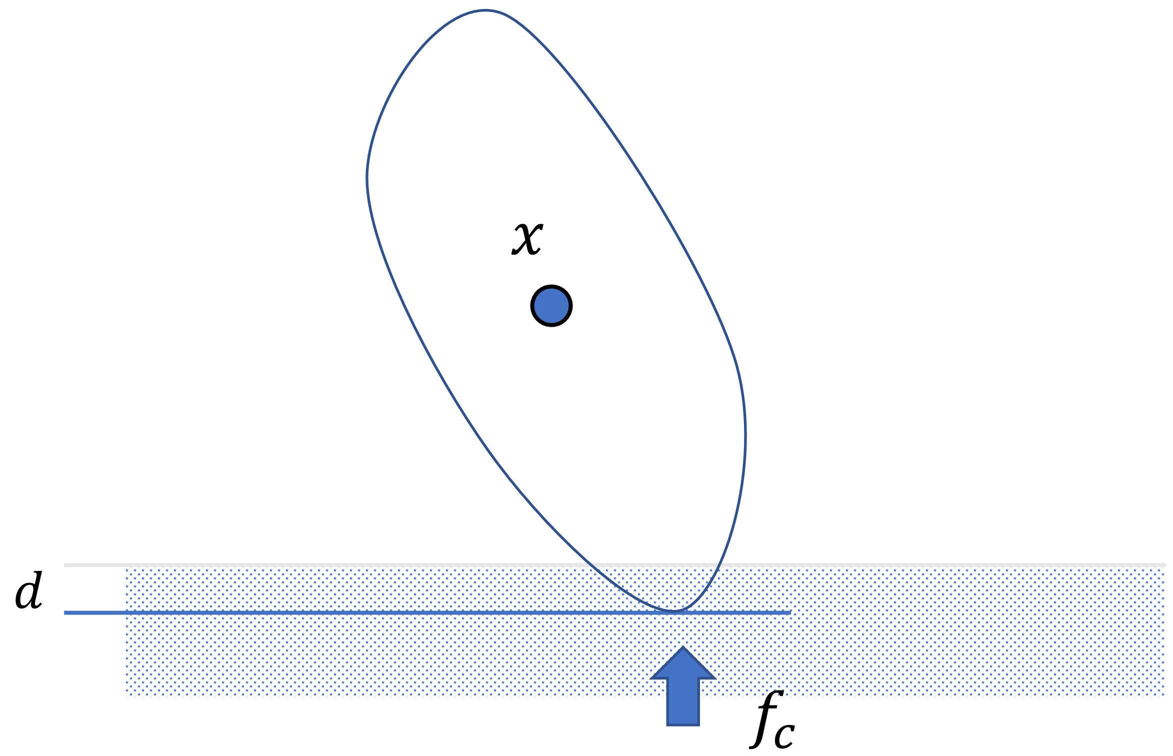
$m_i, I_i, x_i, R_i, v_i, \omega_i$



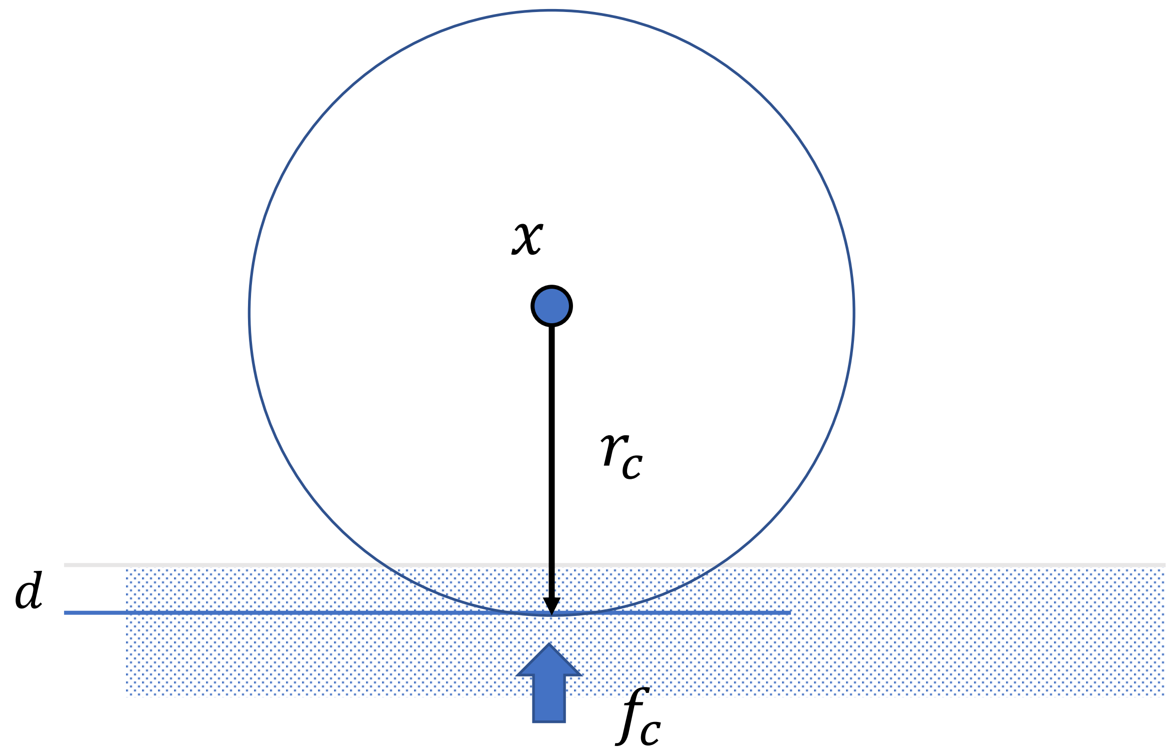
$$M\dot{\boldsymbol{v}} + C(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{f} + \boldsymbol{J}^T\boldsymbol{\lambda}$$

$$J\boldsymbol{v} = 0$$

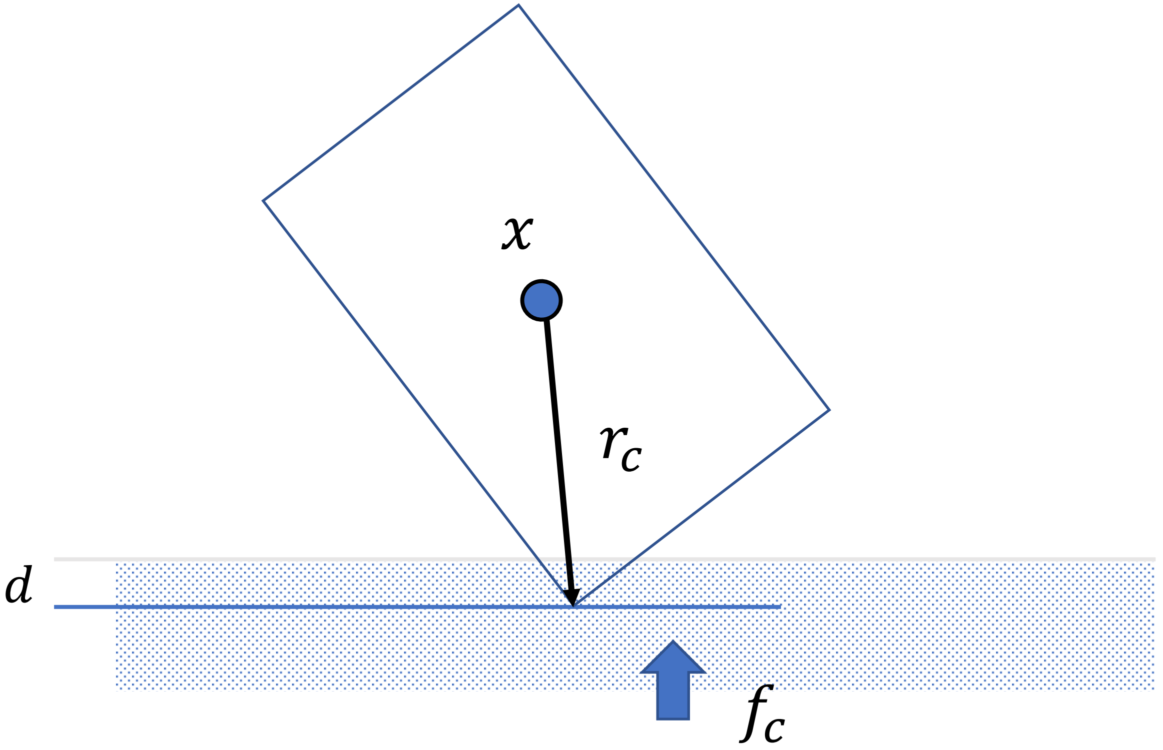
Contacts



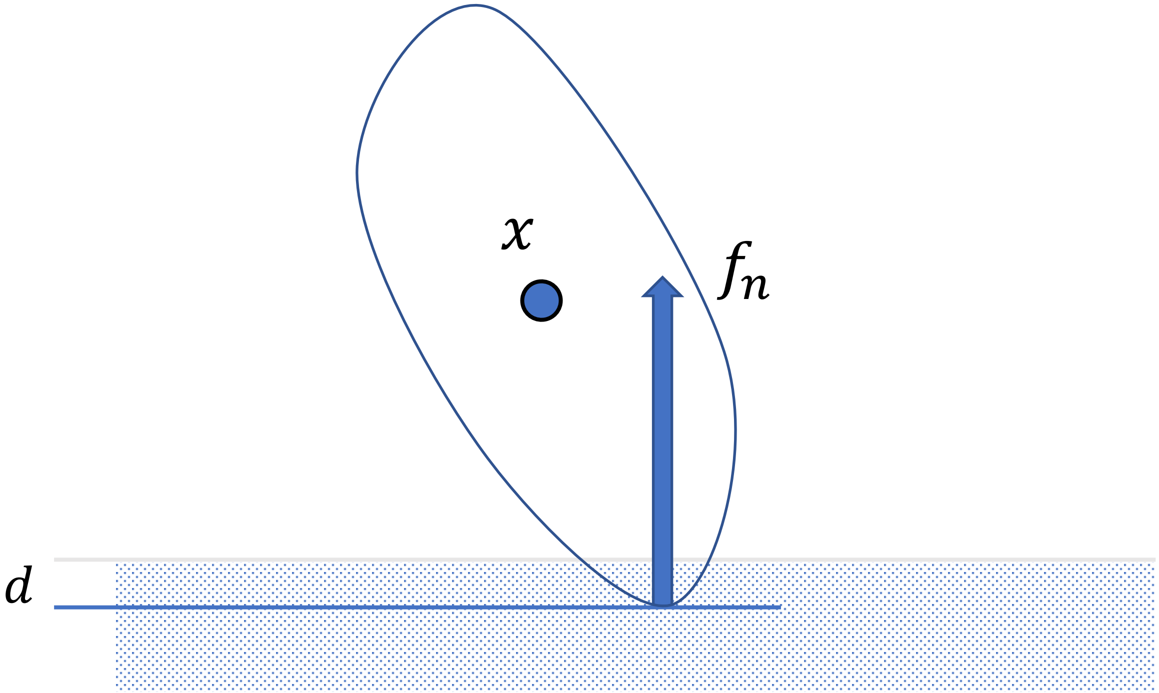
Contact Detection



Contact Detection

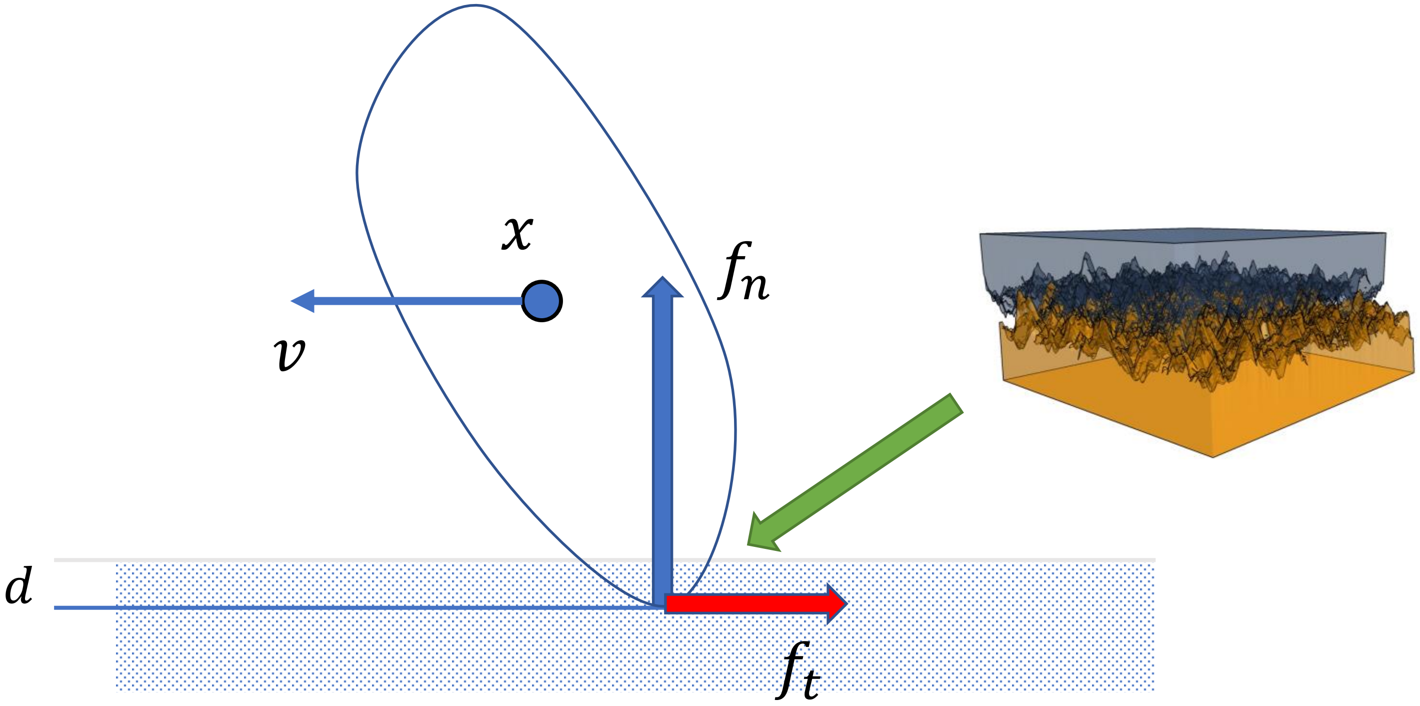


Penalty-based Contact Model



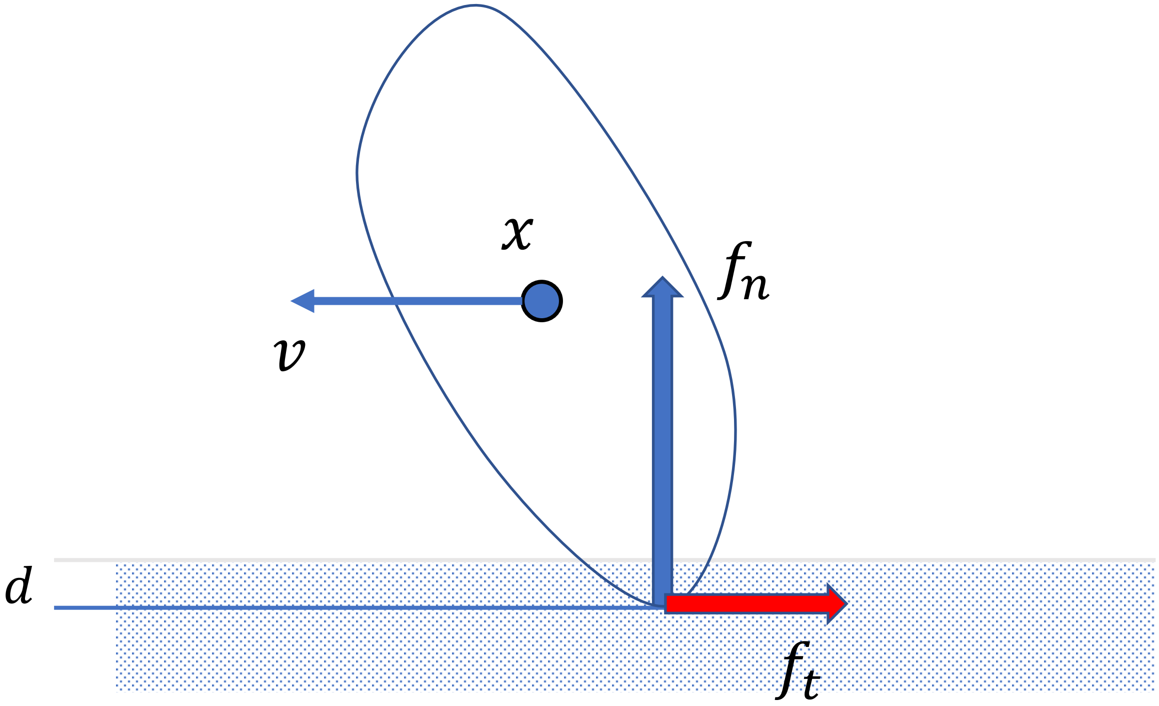
$$f_n = -k_p d - k_d v_{c,\perp}$$

Frictional Contact



Coulomb's law of friction: $|f_t| = \mu f_n$

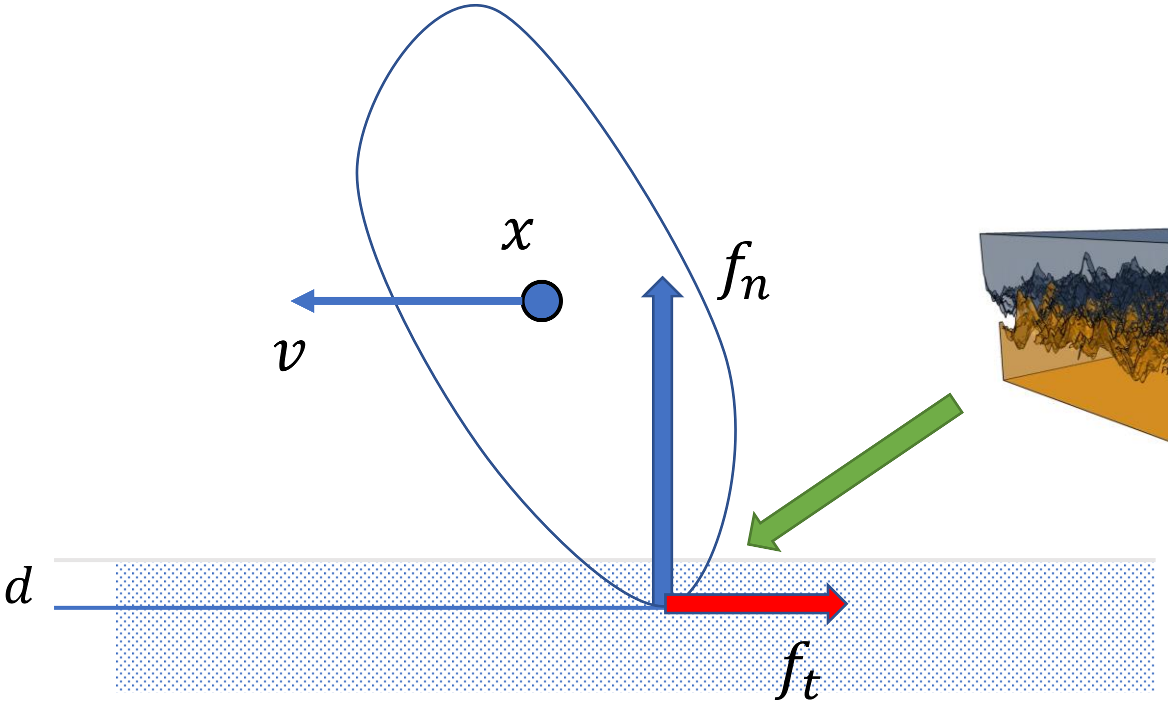
Frictional Contact

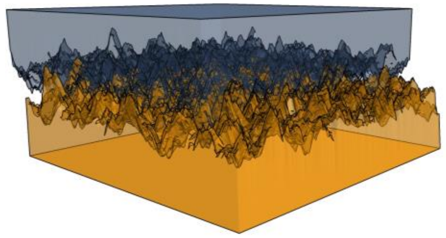


$$f_n = -k_p d - k_d v_{c,\perp}$$

$$f_t = -\mu f_n \frac{v_{c,\parallel}}{\left\| v_{c,\parallel} \right\|}$$

Frictional Contact

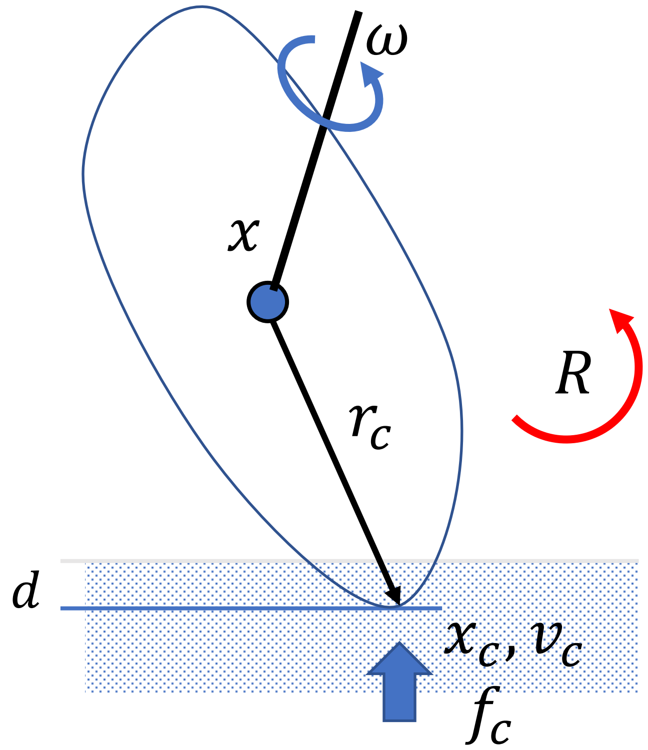




Coulomb's law of friction: $|f_t| \leq \mu f_n$

How to model static friction???

Contact as a Constraint

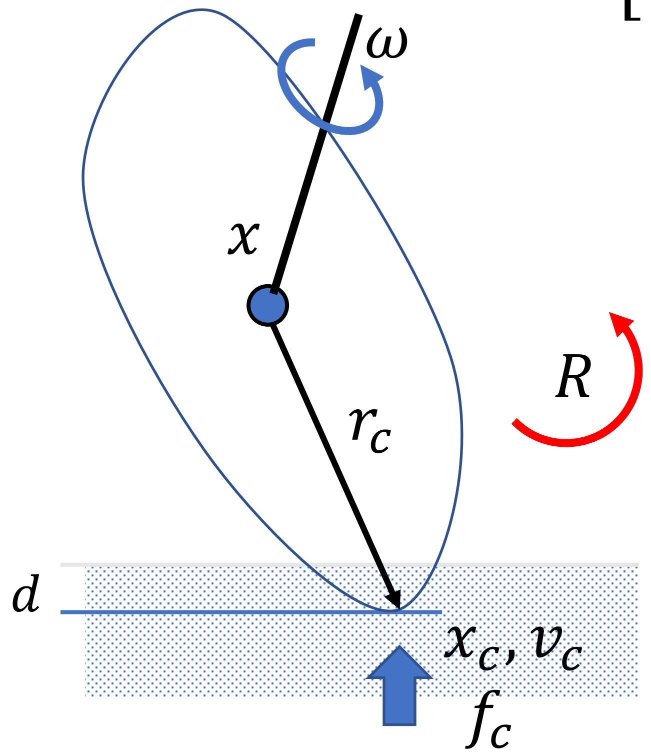


$$x_c = x + r_c$$

$$v_c = v + \omega \times r_c = J_c \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$v_{c,\perp} = v + \omega \times r_c = J_{c,\perp} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Contact as a Constraint

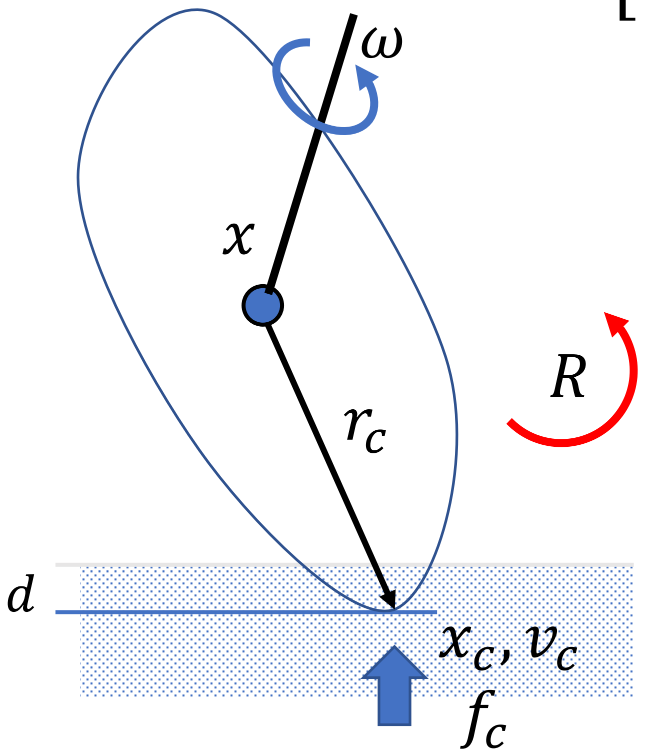


$$\begin{bmatrix} m\mathrm{I}_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega \times I\omega \end{bmatrix} = \begin{bmatrix} f \\ \tau \end{bmatrix} + J_c^T \lambda$$

$$v_c = J_c \begin{bmatrix} v \\ \omega \end{bmatrix} \geq 0$$

$$\lambda \geq 0$$

Contact as a Constraint



$$\begin{bmatrix} m\mathrm{I}_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega \times I\omega \end{bmatrix} = \begin{bmatrix} f \\ \tau \end{bmatrix} + J_c^T \lambda$$

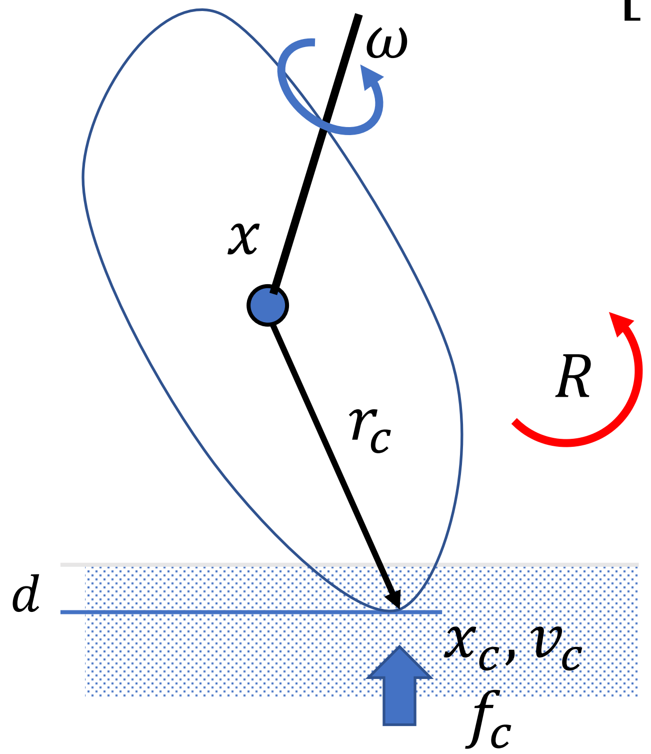
$$v_c = J_c \begin{bmatrix} v \\ \omega \end{bmatrix} \geq 0$$

$$\lambda \geq 0$$

$$v_c > 0 \Rightarrow \lambda = 0$$

$$\lambda > 0 \Rightarrow v_c = 0$$

Contact as a Linear Complementary Problem



$$\begin{bmatrix} m\mathrm{I}_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega \times I\omega \end{bmatrix} = \begin{bmatrix} f \\ \tau \end{bmatrix} + J_c^T \lambda$$

$$v_c = J_c \begin{bmatrix} v \\ \omega \end{bmatrix} \geq 0$$

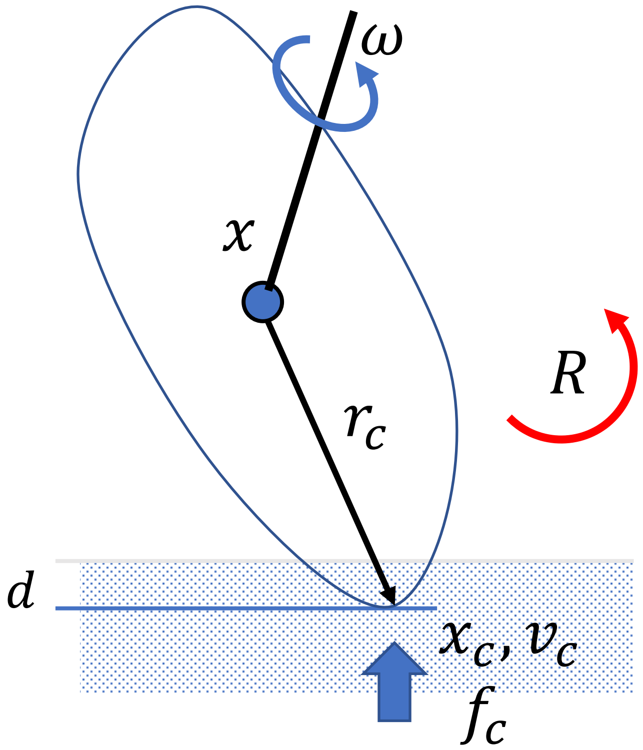
$$\lambda \geq 0$$

$$v_c \perp \lambda = 0$$

(Mixed) Linear Complementary Problem (LCP)

To solve an LCP:
e.g. Lemke's algorithm – a simplex algorithm

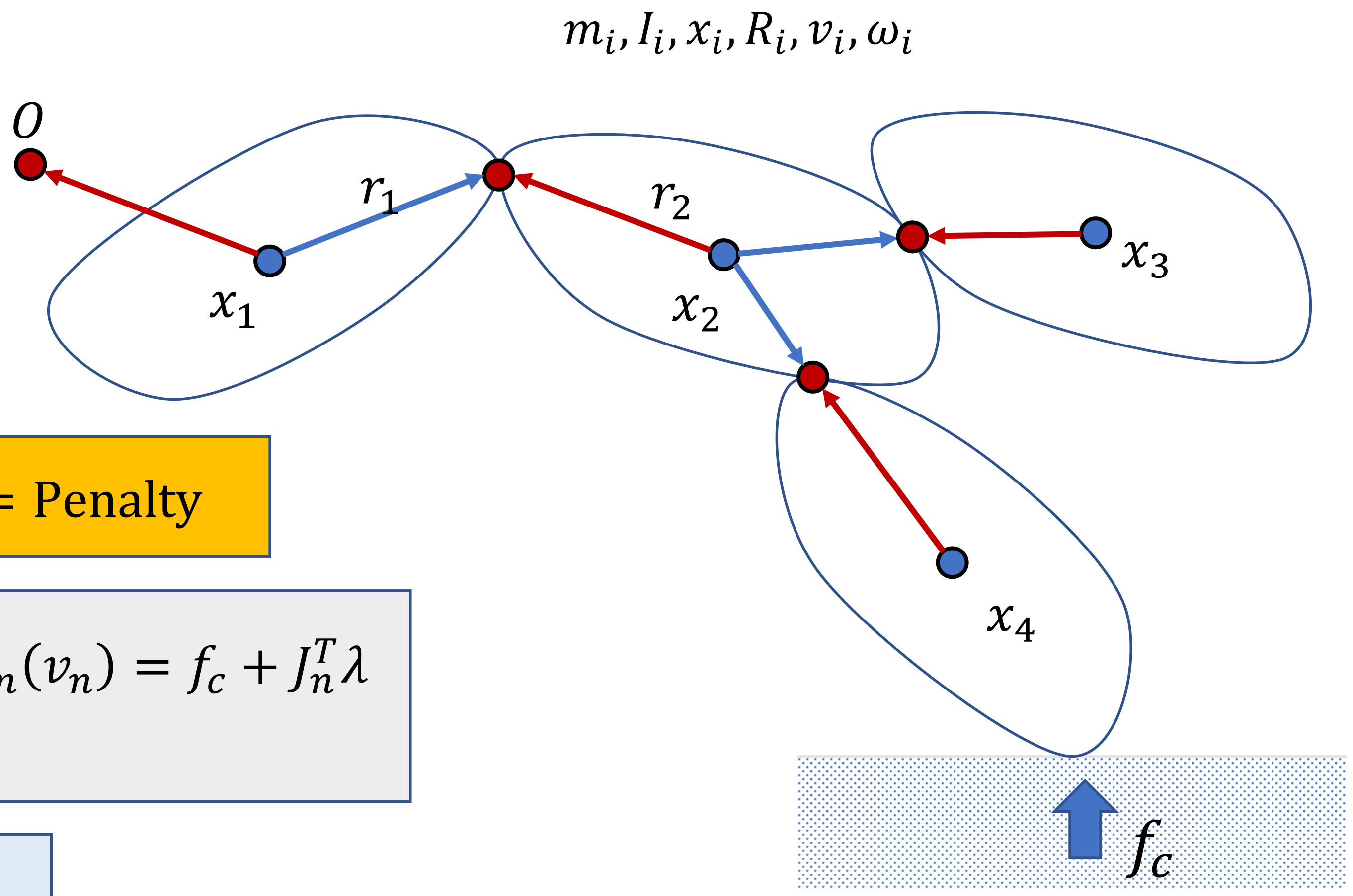
Contact as a Linear Complementary Problem



How to deal the friction?

David Baraff. SIGGRAPH '94
Fast contact force computation for nonpenetrating rigid bodies.

Simulation of a Rigid Body System



$I_n = R_n I_0 R_n^T$

$f_c = \text{Penalty}$

$$M_n(v_{n+1} - v_n)/_h + C_n(v_n) = f_c + J_n^T \lambda$$

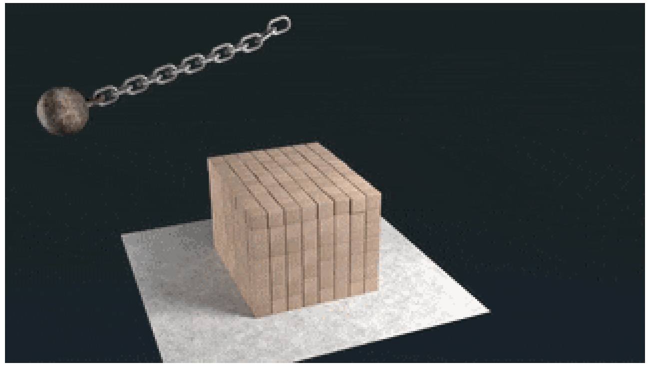
$$J_n v_{n+1} = c_n$$

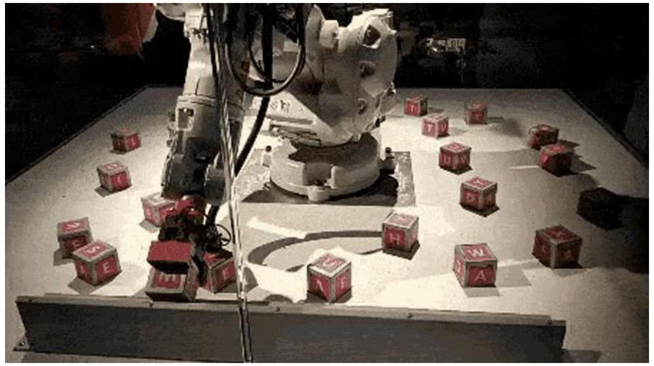
$$x_{n+1} = x_n + h v_{n+1}$$

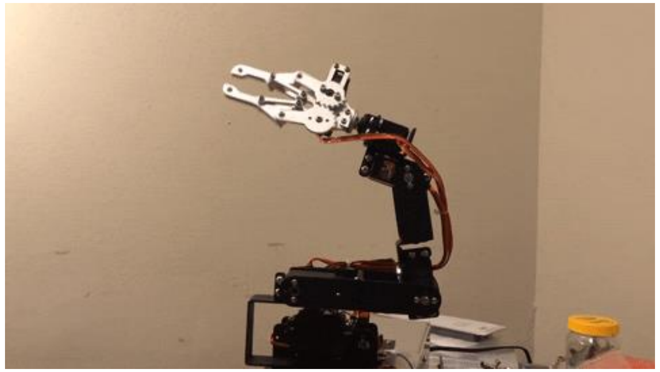
$$q_{n+1} = q_n + \frac{h}{2} \bar{\omega}_{n+1} q$$

Outline

- Simulation Basis
  - Numerical Integration: Euler methods
- Equations of Rigid Bodies
  - Rigid Body Kinematics
  - Newton-Euler equations
- Articulated Rigid Bodies
  - Joints and constraints
- Contact Models
  - Penalty-based contact
  - Constraint-based contact







https://www.cs.cmu.edu/~baraff/sigcourse/

Questions?